Multiunit Dynamic Pricing with Different Types of Observable Customer Information

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Abstract

In various sectors, such as retail, firms encounter customers with multiunit demand and often implement nonlinear pricing to accommodate this demand structure. While effective, this pricing strategy lacks the adaptability offered by dynamic pricing, a trend gaining significance in the retail landscape due to technological advancements. Neglecting multiunit demand in dynamic pricing, however, can result in suboptimal prices and revenue losses. In response, this paper introduces multiunit dynamic pricing which integrates the strengths of both nonlinear and dynamic pricing strategies.

We formulate a stage-wise optimization problem, considering customer preferences for batches of a product through a model based on random willingness-to-pay. The willingness-to-pay is influenced by a combination of the customer's attraction to and consumption of the product—both private information. The firm, functioning as a monopoly, has the ability to price-discriminate between various order sizes by quoting nonlinear batch prices.

Our investigation explores three cases of observable information: attraction to the product, consumption of the product, or both. Optimality conditions are derived for all cases, establishing a closed-form expressions for two of them. Additionally, we demonstrate the preservation of desirable monotonicity in time and capacity. Leveraging this monotonicity, we showcase the dynamics of the optimal pricing policy. A simulation study underscores the potential of our approach, highlighting the value of information in supporting strategic decisions, particularly regarding investments in customer profiling and segmentation. Furthermore, we illustrate how our solutions enable firms to make informed stocking and restocking decisions, providing practical insights for firms in multiunit dynamic pricing environments.

Keywords: Revenue Management, Dynamic Pricing, Nonlinear Pricing, Multiunit Purchases, Customer Choice

1 Introduction

Nonlinear pricing has been a widespread practice in many industries, particularly in retail, for quite some time. The objective of this pricing strategy is to increase overall demand by tempting customers to buy more. Examples of nonlinear pricing include special offers such as "buy 3, pay 2" or "take an additional item, get 25% off," as well as volume discounts where customers pay lower unit prices for purchasing more units.

Dynamic pricing, on the other hand, is a relatively new field but its significance has been growing in recent years, particularly with the emergence of e-commerce and digital price tags in physical stores. With the capability of quickly adjusting prices, dynamic pricing has had a significant impact on various industries such as travel, hospitality, entertainment, electricity, and retail. Through e-commerce platforms and loyalty cards, sellers have access to more information about a customer's purchasing behavior which, combined with the ability to adjust prices, can significantly influence a seller's earnings.

Standard dynamic pricing assumes that customers will buy only one unit at a time. While this assumption may be reasonable in some cases (such as car rental or hotel rooms), it is not applicable in many other situations, such as for most grocery and fashion products. Neglecting the possibility to influence customers' purchase quantity can lead to suboptimal prices and lost revenue in these cases. To fully leverage the revenue potential in such fields, a combination of nonlinear and dynamic pricing is highly desirable.

This paper addresses precisely such a scenario by introducing nonlinear prices in a multiunit dynamic pricing setting. Here, customers are assumed to have multiunit demand and the product is available for purchase in all batch sizes, ranging from a single unit to the entirety of the remaining stock. The purchase quantity is influenced by a nonlinear pricing scheme, deviating from the traditional approach of quoting a single unit price with batch prices derived from multiplying batch size by unit price. The objective is to dynamically quote batch prices for a single product to maximize expected revenue. The selling horizon and product inventory are limited, and after prices are quoted for each batch size, customers purchase one of these batches or nothing at all.

Our model assumes customers hold an undisclosed willingness-to-pay for each batch size, optimizing their utility by choosing the batch size with the greatest surplus over the quoted price. To capture this, we use a two-parameter approach – integrating a base willingness-to-pay reflecting interest and a consumption indicator signaling diminishing marginal appreciation. Both parameters are modeled as random variables. Additionally, we explore scenarios where the firm gains insight into the next customer's choice parameters, observing their base willingness-to-pay, consumption indicator, or both before quoting batch prices. This mirrors practices in personalized online pricing, profiling logged-in customers based on purchase history. For non-logged-in customers, technologies like applets and cookies facilitate customer profiling (see e.g., Raghu et al., 2001).

We contribute to the multiunit dynamic pricing literature through a stochastic dynamic optimization model. This model quotes batch prices, influencing random customer demand to maximize expected revenue. We consider various types of observable information about customer choice parameters, adapting the model for each type and solving it analytically. Our study reveals key properties of the value function and optimal batch prices. Notably, we prove the monotonicity of expected revenues with respect to capacity and time, in the context of multiunit purchases. This property aligns with an intuitive understanding of pricing dynamics relative to product scarcity. Importantly, the monotonicity in capacity ensures a unique optimal solution in our stage-wise optimization.

In our simulation study, we examine the value of information by comparing the three types of observation. Additionally, we consider a scenario where the firm lacks the ability to observe customers' choice parameters. For this situation, Schur (2023) proposed a heuristic solution mechanism, which we briefly explain and apply. Furthermore, we assess the impact of distribution on expected revenues, assuming both uniform and normal distributions. In another study, we relax the assumption of precise customer information observation. Instead, we allow the firm to accurately assign customers to predefined segments, narrowing down the distribution of corresponding random variables. Lastly, we introduce an additional layer of decision-making: stocking and restocking.

The implications of our work can be summarized as follows:

- When the firm accurately observes one or both of customers' choice parameters, our models offer optimal batch prices for every state in the selling horizon. Moreover, knowing the optimal expected revenue for every stocking level enables the firm to make optimal stocking and restocking decision.
- Understanding the value of information allows firms to evaluate the profitability of potential investments in customer profiling or segmenting, contributing to strategic decision-making in the ever-evolving landscape of dynamic pricing.
- Our value function serves as an upper bound in restricted pricing scenarios, aiding firms in assessing potential revenue losses from pricing structure limitations (e.g., linear price with volume discounts like “3 for 2”).
- In settings where customer parameters are unobservable, our findings offer valuable structural insights. These insights can serve as the foundation for effective heuristics, ranging from simple business rules (e.g., "additional units at a 10% discount") to more sophisticated strategies (refer to Schur, 2023).

This paper is organized as follows: In Section 2, we give a short overview of existing literature connected to multiunit dynamic pricing. We then present the setting, the customer choice as well as a general optimization model in Section 3. In Section 4, we present three adjustments to the general model to deal with three types of observations regarding customer choice parameters. In Section 5, we conduct our simulation study.

2 Literature review

In this paper, we extend dynamic pricing with nonlinear pricing. Accordingly, we start by shortly reviewing literature from both streams, nonlinear pricing and dynamic pricing. Thereafter, we focus on research belonging to multiunit or multiproduct dynamic pricing. The first category also covers the (scarce) literature on nonlinear dynamic pricing (as nonlinear pricing requires multiunit demand). The second category is primarily related to our setting because of the applied customer choice models where customer choose one of several options.

Nonlinear pricing is an often-applied pricing scheme that can be found in many industries including e.g. telecommunications, transportation, energy, supply chains, and retail. This broadness results in a diverse body of literature and is addressed by Wilson (1993) by giving an overview of application, substantial economics and marketing. Most nonlinear pricing research from the economics literature considers only static pricing. This can be observed in the review articles of Stole (2007) and Armstrong (2016) where only a small portion of covered literature assumes a dynamic environment. This literature commonly assumes these dynamics stem from competing firms and lock-in effects of recurring customers. More relevant to our setting is literature that focus on a dynamic environment stemming from dynamic demand (e.g., Dhebar & Oren, 1986, and Braden & Oren, 1994). However, different to our setting, this research does not consider a product with limited stock which is one of the core assumptions in most of dynamic pricing literature (cf., e.g., Talluri & van Ryzin, 2004 (Chapter 5) and Phillips, 2005 (Chapter 10)).

In the following, we turn our attention on dynamic pricing of a product with limited stock, finite selling horizon, and customer choice behavior. One of the first to consider such a setting were Gallego and van Ryzin (1994) who showed that the optimal price increases with remaining time and decreases with remaining stock. Their work laid the foundation for dynamic pricing as an emerging discipline in revenue management, which, during that period, was predominantly influenced by capacity control. Afterwards, dynamic pricing gained a lot of attention by researchers. They often focused on finding optimality conditions and showing monotonic behavior in time and capacity. This research was summarized by several review articles (e.g., Bitran & Caldentey, 2003, Chiang et al., 2007, and, with a special focus, Gönsch et al., 2013 and den Boer, 2015) as well as textbooks (e.g. Talluri & van Ryzin, 2004 (Chapter 5) and Phillips, 2005 (Chapter 10)).

By dropping the common assumption that customers purchase at most one unit of a product, a new stream in the dynamic pricing community was born. Multiunit dynamic pricing considers multiunit demand, which, in turn, is the basis for combining nonlinear and dynamic pricing. An early publication in this stream is Elmaghraby et al. (2008) where the optimal design of a markdown pricing mechanism in the presence of multiunit demand was analyzed. They assume a full information setting meaning that the firm knows at the beginning of the selling horizon every customer and the respective values of their willingness-to-pay. This setting aligns to one of the three scenarios in our study (refer to Section 4.3). In this scenario, we assume full information availability regarding the current customer. In contrast, the other two scenarios (outlined in Sections 4.1 and 4.2) involve customer decisions based on private, rendering them unpredictable in advance. Furthermore, future revenues remain uncertain across all scenarios. Thereby, we acknowledge that firms typically cannot accurately predict specific customer streams or their purchasing behavior with absolute certainty. Levin et al. (2014) introduce a dynamic pricing model with stochastic batch demand. They assume customers have a certain batch size they want to purchase and request exactly and only this batch size from the firm. The firm then quotes a price and customers either buy the batch or leave the shop without purchasing anything. The authors show optimality conditions and prove monotonic properties of optimal policy and value function. However, their setting does not accommodate nonlinear pricing as customers exhibit inflexibility in their purchase quantities (such as a family buying flight tickets for their vacation), rendering firms unable to influence these quantities through the application of appropriate nonlinear batch prices. In our study, we presume customers to be flexible concerning batch sizes, as is common in retail scenarios. Instead of specifying a particular batch size, customers observe quoted nonlinear prices for various batch sizes and select the one that maximizes their utility. This flexibility introduces complexity to the optimization model, as firms must decide on several prices simultaneously while anticipating a broader range of potential customer reactions.

There are currently only two other research articles that consider stochastic flexible multiunit demand that can be influenced by nonlinear dynamic pricing: Gallego et al. (2020) and Schur (2023). Gallego et al. (2020) consider three dynamic pricing models: nonlinear, linear, and block pricing. They consider utility maximization choice models where customers are characterized by one single parameter. This parameter cannot be observed by the firm and is modeled as random variable. The authors give optimality conditions for their nonlinear dynamic pricing model and show structural properties like the monotonicity in time and inventory. Their work is related to this paper in the following way: Our scenarios where a firm can observe one of two choice parameters is an extension to a special case of their nonlinear pricing model. The key distinction in our approach lies in our customer choice model, where customers are characterized by two parameters: one reflecting the product's attractiveness and another indicating the inclination to purchase multiple units. This allows for nuanced variations among customers. For instance, a parent may value diapers more than a childless individual, and a family might be more prone to buying several packs of toilet paper compared to someone living alone. Schur (2023) considers the same customer choice model as we do, but without the possibility to (partially) observe customers' private information. In the absence of such information, the optimization model becomes analytically intractable, leading Schur (2023) to develop three heuristics. These heuristics, utilizing fluid approximation, are designed to be asymptotically optimal. In our setting, where we assume partial observation of current customer information, we encounter distinct yet related optimization problems that can be solved to optimality (numerically). Additionally, we demonstrate that the well-known monotonicity in time and inventory persists in our scenario. Lastly, our simulation study explores the value of knowing customers' private information, contributing to the understanding of the situational and contextual value of different types of information.

Multiproduct dynamic pricing is another field in the domain of dynamic pricing that emerged more and more in recent years (see for a review, e.g., Chen & Chen, 2015). By defining batches of a single product as several different products, we can draw a parallel between multiproduct and multiunit dynamic pricing. In multiproduct dynamic pricing the products often are substitutes and an upcoming customer can pick at most one of these products. Customers' demand is stochastic and the firm is facing a finite selling horizon with scarce product-dependent inventory (see, e.g., Zhang & Cooper, 2009, Dong et al., 2009, and Akçay et al., 2010). The main difference between multiunit (i.e., our work) and these multiproduct dynamic pricing models is the inventory structure. Whereas every product has its own inventory in multiproduct dynamic pricing, every batch ("product") exploits the same inventory (but in another quantity) in multiunit dynamic pricing. One exception to the product-specific inventory setting is Maglaras and Meissner (2006). They analyze a setting where every product consumes one unit of the same resource. This assumption leads to different pricing dynamics when compared to our setting, where each product has varying resource consumption based on batch size. Consequently, in our context, each batch price reacts differently to changes in dynamic scarcity, unlike their setting where all products equally respond to the dynamic scarcity of the common resource. Maglaras and Meissner (2006) show that dynamic pricing and capacity control can be reduced to a common formulation. Instead of concentrating on dynamic pricing or capacity control, the firm finds optimal decisions by controlling the consumption rate of every product regarding resource capacity.

In our literature review, it becomes evident that research on nonlinear dynamic pricing is exceptionally limited, with only two notable exceptions: Gallego et al. (2020) and Schur (2023). However, these works have distinctive characteristics that set them apart. Gallego et al. (2020) focuses on a model where customer behavior is characterized by a single (random) parameter, whereas our approach involves two (random) parameters. This enables us to capture a more individualized customer choice behavior and introduces additional uncertainty into the optimization problem. On the contrary, Schur (2023) employ the same customer choice model as we do. However, different from our setting, they cannot observe customers' private information. With these observations, we (numerically) solve the optimization model to optimality and determine the value of information in a simulation study. Notably, other works in the field diverge significantly in at least one critical assumption, leading them to analyze distinct settings. In many cases, these works do not consider customers with flexible multiunit demand, and consequently, do not explore the application of nonlinear pricing schemes to influence stochastic purchase quantities.

3 Problem definition

After introducing general setting and notation in Section 3.1, we present the customer choice model in Section 3.2. Building on this, we finally introduce the optimization model in Section 3.3.

3.1 General setting and notation

We introduce the following framework to combine nonlinear and dynamic pricing: A monopolistic firm sells a single product with fixed stock C over a finite selling horizon with T periods. The selling horizon is indexed backwards in time, i.e., periods T and 0 mark the beginning and the end, respectively. We assume that exactly one customer arrives in each period $t = T, T-1, \ldots, 1$ and is interested in buying one or more units of the product depending on the batch prices the firm is quoting. The capacity of the product is nonreplenishable and any capacity left at the end of the selling horizon ($t = 0$) is worthless to the firm. At any point in time t, the firm decides on batch prices $\boldsymbol{r} = (r_1, r_2, \ldots, r_c)^T$ based on remaining capacity $c \leq C$ and expectations of future demand. Thereby, the remaining capacity c defines the maximal possible batch size j that could be offered. Each r_j represents the price a customer must pay for a batch of j units. The firm's goal is to set the prices that maximize overall revenue, taking into account future demand and customer behavior. Arriving customers react on quoted batch prices and decide on the batch size to purchase, with $p_j(\boldsymbol{r})$ denoting the probability that an arriving customer chooses to buy j units. In this case, the firm immediately earns r_j in revenue and product's capacity is lowered by j. Throughout the remainder of this paper, to simplify our notation, we adopt the convention that $r_0 = 0$.

3.2 Customer Choice Model

In our setting, customers face several options (i.e., batch sizes, including also a batch of zero) and pick exactly one of these. We assume that customers have a personal (unknown to the firm) evaluation for each option and this evaluation can be expressed monetarily via customers' willingness-to-pay. The utility, representing the difference between customers' willingness-to-pay and the price, determines the choice, with customers opting for the option that yields the highest utility. This model is commonly employed in economic and pricing literature as it captures customers heterogeneity regarding their preferences (via personal willingness-to-pay) and firm's influence on customers' decision (via price). Moreover, it relies on a sound theoretical groundwork, as it aligns with economic principles and the rationale that individuals make decisions based on perceived value and cost considerations. Specifically, if customers face multiple options rather than a binary decision (such as purchasing or not), this model is often applied (see, e.g., Braden and Oren (1994) with their nonlinear (static) pricing setting, and Akçay et al. (2010) with their multiproduct dynamic pricing setting).

Customers' willingness-to-pay X_j for a batch of size j is private information and unknown to the firm. This makes X_j a random variable and a proper model is needed to reflect customers' preferences. In literature, a common assumption is that marginal willingness-to-pay, i.e., $X_{j+1} - X_j$, is non-negative and decreasing (see, e.g., Baucells & Sarin, 2007, Goldman et al., 1984, Iyengar & Jedidi, 2012, and Gallego et al., 2020). This assumption translates to: "An additional unit is never bad, but it is less appreciated than the previous one." There are several methods to model random willingness-to-pay in settings where customers buy in batches. The model we apply is based on a formulation of Iyengar and Jedidi (2012) and was also applied by Schur (2023). Iyengar and Jedidi (2012) introduce a willingness-to-pay function that depends on known parameters. Uncertainty regarding customers' behavior is then added with the help of an error term. Schur (2023) adapt this willingness-to-pay function. However, instead of using known parameters and adding randomness via an error term, the parameters itself are assumed to be private information, and thus, depicted by random variables. We follow the latter approach and define the willingness-to-pay X_j for a batch size of j by:

$$X_j = \omega \cdot \sum_{k=0}^{j-1} (\lambda)^k \qquad \text{for } j = 1, \dots, c, \qquad (1)$$

with independent continuous random variables ω and λ. We denote the corresponding density functions by f_ω and f_λ. Likewise, the cumulative distribution functions are given by F_ω and F_λ. We assume the support of both density functions is $[0, 1]$. Furthermore, we make an assumption regarding the continuous failure rates of both random variables, ω and λ, defined over the interval $(0, 1]$ by $h_\omega(x) = \frac{f_\omega(x)}{1-F_\omega(x)}$ and $h_\lambda(x) = \frac{f_\lambda(x)}{1-F_\lambda(x)}$, respectively. We assume that these failure rates are increasing in x. This assumption ensures the existence of a unique solution to our optimization problem, as evidenced by the proof of Propositions 1 and 5. This is consistent with common practices in the standard literature, as random variables with increasing failure rates have an increasing generalized failure rate (see Lariviere, 2006). It aligns with one of the three standard assumptions mentioned in Ziya et al. (2004). Furthermore, it is compatible with numerous probability distributions, including but not limited to the uniform, triangular, normal, exponential, Weibull, Gumbel, gamma distributions, and their truncated variants (some of them with restrictions regarding parameter choice) (see Banciu & Mirchandani, 2013).

By restricting λ on $[0, 1]$, we assure that marginal willingness-to-pay, i.e., $X_{j+1} - X_j = \omega \cdot \lambda^j$, is non-negative and decreases in quantity j (given $\omega \geq 0$). Thereby, this model covers the common assumption regarding customers' preferences that was stated earlier in this section. Restricting ω on $[0, 1]$ is only a matter of scaling and normalizes marginal willingness-to-pay. The interpretation of the random variables, ω and λ, is the following: As ω equals $X_1 = \omega \cdot \sum_{k=0}^{0} (\lambda)^k = \omega$ and influences $X_j = \omega \cdot \left(\sum_{k=0}^{j-1} (\lambda)^k\right)$, $j \geq 2$, in a linear manner, we can interpret it as attractiveness of the product to the customer. We call this parameter base willingness-to-pay. In contrast, the consumption indicator λ has no influence on $X_1 = \omega$, but depicts the rate at which marginal willingness-to-pay is diminishing in j. This can be observed by $X_{j+1} - X_j = \omega \cdot \lambda^j = \lambda \cdot \left(\omega \cdot \lambda^{j-1}\right) = \lambda \cdot \left(X_j - X_{j-1}\right)$. We can interpret λ as customers' willingness to stockpile or consume.

The following figure provides an illustrative representation of willingness-to-pay curves for three specific customers, each characterized by unique realizations of random variables ω and λ, denoted as w and l, respectively.

In this example, customer 1 (dashed line) shares the same base willingness-to-pay ($w = 0.8$) with customer 2 (dotted line), and the same consumption indicator ($l = 0.8$) with customer 3 (solid line). Consequently, customers 1 and 2 exhibit identical willingness-to-pay values for a batch size of 1, implying they have a similar valuation of the product. However, a notable distinction arises when we examine the curves further. While the solid curve steadily increases until $j = 10$, the dotted curve reaches a relatively constant level at $j = 4$. This divergence stems from the fact that customer 1, with a consumption indicator twice as high, is significantly more interested in purchasing larger batches compared to customer 2.

Comparing customer 1 and 3, we observe that the solid line consistently falls exactly between the dashed line and zero. This is a direct consequence of both customers having the same consumption indicator, but with customer 3 having only half the base willingness-to-pay of customer 1. As a result, customer 1 is willing to pay twice as much as customer 3, indicating a substantially higher appreciation for the product.

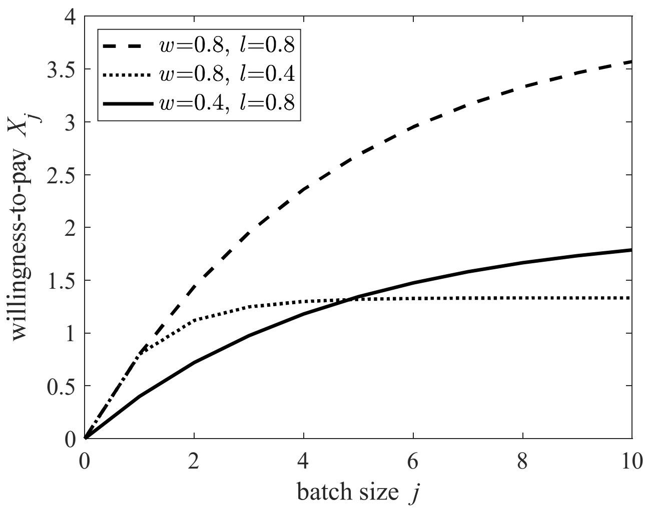

Figure 1: Three exemplary willingness-to-pay curves for $j \leq 10$

From a theoretical perspective, if the firm were given the choice among the three customers, it would naturally prefer to serve customer 1, as it can charge the highest prices for each batch size. However, when deciding between customer 2 and 3, the choice is less clear-cut. When facing a stock shortage, serving customer 2 might be preferable, while in situations with ample stock availability, customer 3 could be the better option.

Briefly leaving the example behind us allows for the definition of customers' utility. The utility $u_j(\boldsymbol{r})$ for purchasing j units is the difference between their willingness-to-pay X_j and price r_j:

$$u_j(\boldsymbol{r}) = X_j - r_j \qquad \text{for } j = 1, \dots, c. \tag{2}$$

Customers act rational and choose the option that yields the highest utility. Thus, they purchase j units if and only if $u_j(\boldsymbol{r}) = \max_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}$ with $u_0(\boldsymbol{r}) = 0$ denoting the no-purchase option.

Resuming the previous example (Figure 1), we introduce an arbitrary price vector (as shown by the red line in Figure 2, left side). It's important to highlight that while we use a linear pricing scheme in this particular illustration, our model is not restricted to linear pricing and explicitly accommodates non-linear pricing structures. The application of equation (2) results in the generation of three distinct utility curves (depicted in. Figure 2, right side), one for each customer.

Upon close examination, we can observe that customer 1 (dashed line) has maximal utility at $j = 3$, customer 2 (dotted line) at $j = 1$, and customer 3 (solid line) at $j = 0$. In a scenario where these three customers collectively constitute the entire market and each customer's arrival is equally likely, the firm would have the following probabilities of selling units with this price vector: 0 units, 1 unit, or 3 units, each with a probability of $\frac{1}{3}$.

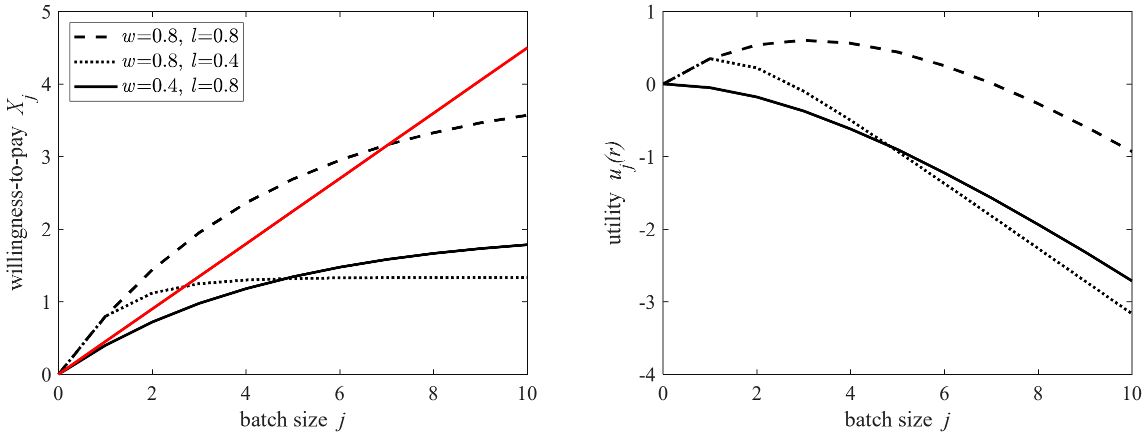

Figure 2: Example of Figure 1 with added price curve (left) and resulting utilities (right)

Given our assumption that ω and λ are continuous random variables, we find ourselves in a realm with an infinite number of willingness-to-pay curves, each representing a specific customer. In this expansive landscape, it is impractical to individually assess every customer to pinpoint where their maximum utility lies, as we did in the example. Instead, when presented with a specific price vector $\boldsymbol{r}$, we want to determine which utility curves, described as combinations of w and l, have their maximum at batch size j. In essence, for any given j, we seek all (w, l) pairs for which $u_j(\boldsymbol{r}) = \max_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}$. According to equation (2), this condition holds for all (w, l) that satisfy:

$$w \cdot \sum_{k=0}^{j-1} l^k - r_j \geq w \cdot \sum_{k=0}^{i-1} l^k - r_i \qquad \text{for } i = 1, \dots, c \text{ and}$$

$$w \cdot \sum_{k=0}^{j-1} l^k - r_j \geq 0. \tag{3}$$

To compute the probability that the next utility curve we encounter attains its maximum at j, we must calculate the probability that (w, l) meets these conditions. This can be achieved using the density functions f_ω and f_λ in combination with an indicator function $1_{\left\{u_j(\boldsymbol{r})=\max\limits_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}\right\}}(w,l)$. This indicator function equals 1 when the condition is met and 0 otherwise. Notably, this probability is technically equivalent to the probability $p_j(\boldsymbol{r})$ of selling j units for a given price $\boldsymbol{r}$ and we can express it as:

$$p_j(\boldsymbol{r}) = \int_0^1 \int_0^1 f_\omega(w) f_\lambda(l) 1_{\left\{u_j(\boldsymbol{r})=\max\limits_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}\right\}}(w,l)\, dw\, dl \qquad \text{for } j = 1, \dots, c. \tag{4}$$

3.3 Dynamic programming formulation

A firm maximizes expected revenue over the whole selling horizon by solving a dynamic optimization problem. Thereby, it searches for the optimal batch prices r_j, $1 \leq j \leq c$, to offer at every time t with remaining capacity c. The maximal number of purchasable units equals the remaining capacity in every state (t, c). To take the varying character of remaining capacity into account, we define a state-dependent action space $\mathcal{R}_c = \{\boldsymbol{r} \in \mathbb{R}^c : r_j \geq 0, j = 1, \dots, c\}$ with $\mathcal{R}_0 = \emptyset$. Action space $\mathcal{R}_c$ defines the set of feasible solutions to our maximization problem. By taking the remaining capacity c into account, it makes sure that only available batch sizes $j \leq c$ are offered. The dynamic problem is given by:

$$V_t(c) = \max_{\boldsymbol{r} \in \mathcal{R}_c} \left\{ \sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j + V_{t-1}(c-j)\right) + \left(1 - \sum_{j=1}^{c} p_j(\boldsymbol{r})\right) \cdot V_{t-1}(c) \right\} \tag{5}$$

where $V_t(c)$ denotes the optimal expected revenue-to-go from period t onwards with remaining capacity c. The boundary conditions are $V_0(c) = 0$ for $c \geq 0$ and $V_t(0) = 0$ for $t \geq 0$.

In every state, one out of $c + 1$ random events occurs: A customer purchases $0 \leq j \leq c$ units at a price of r_j with probability $p_j(\boldsymbol{r})$. Additionally, the firm can expect future revenues from remaining capacity $c - j$ and time $t - 1$. We denote the optimal batch prices selected in a state (t, c) by $\boldsymbol{r}_t(c) \in \mathcal{R}_c$.

An alternative formulation of (5) focuses on opportunity costs regarding selling j units, i.e.

$$\Delta_j V_t(c) = V_t(c) - V_t(c-j) \qquad \text{for } j = 1, \dots, c, \tag{6}$$

and is given by

$$V_t(c) = \max_{\boldsymbol{r} \in \mathcal{R}_c} \left\{ \sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right) \right\} + V_{t-1}(c). \tag{7}$$

Thus, the goal to maximize expected revenues can be achieved by maximizing additional revenue gains that are realized by selling up to c units in period t instead of retaining the capacity for later customers. This formulation offers several advantages over (5). The first and most apparent advantage is the immediate insight that optimal prices should surpass opportunity costs. Failing to do so would result in no gain in expected revenue by selling, or worse, it could even lead to a net loss in overall expected revenue. Another advantage becomes evident in later sections as we establish key properties based on formulation (7). These properties are crucial in our pursuit to find the optimal solution of our optimization problem. Lastly, it underscores the significance of opportunity costs, which constitute the sole state-dependent component and are the primary driver behind the dynamic changes in optimal prices over time.

4 Different types of observable information

In this section, we consider different degrees of observability regarding next customer's private information, i.e. base willingness-to-pay ω and consumption indicator λ. In three subsections, we assume that the firm knows at customer's arrival the exact value of base willingness-to-pay, consumption indicator, or both parameters, respectively. Each of these subsections shows the adapted problem formulation, structural properties, and optimal solution (or at least a sufficient condition for optimality).

4.1 Observable base willingness-to-pay

We now consider the case where a firm can observe the base willingness-to-pay of the next customer in line, i.e. the realization w of random variable ω becomes known at the moment the firm decides upon the next batch prices. Consumption indicator λ remains stochastic. Thereby, we eliminate some but not all of uncertainty regarding customers' behavior.

4.1.1. Customer choice and model formulation

Selling at least one unit of the product is now a deterministic occurrence. Notably, for $r_1 < w$, we know for certain that a customer has a higher utility for purchasing one unit than for purchasing nothing at all ($u_1(\boldsymbol{r}) = w - r_1$ is deterministic and positive). However, we still face uncertainty regarding the precise number of units purchased, as we do not know if there are $u_j(\boldsymbol{r})$ values exceeding $u_1(\boldsymbol{r})$.

A customer is indifferent between purchasing zero and one unit of the product when $r_1 = w$. As a tiebreaker, a firm could quote a price that is slightly above or below w (w^+ and w^-, respectively), depending on which outcome would be more suitable. Taking these two strategies explicitly into account would result in increased complexity of notation without adding to understandability. In most instances, a firm prefers customers to purchase at price w. Consequently, we will assume w to act as w^- without further mention. However, there are situations where the firm may not want to sell at w (e.g., if w is too low). In such cases, we will explicitly indicate that the firm employs w^+. Moreover, we ignore the case where a customer might have $w = 0$. This case almost surely does not occur (recall that ω is continuously distributed), and even if it were to occur, it would have no impact. For a customer with a willingness-to-pay of zero for every batch size (as per equation. (1)), there would be no price at which the customer desires to buy while the firm wishes to sell simultaneously. Consequently, the optimal solution in this case would be not to sell anything to that customer.

Observing realization w has the advantage that we can formulate necessary conditions for a customer to purchase j units:

a) $\sum_{k=0}^{j-1} \lambda^k \geq \frac{r_j}{w}$,

b) $\sum_{k=i}^{j-1} \lambda^k \geq \frac{r_j - r_i}{w}$ for all $i \in \{1, 2, \dots, j-1\}$, and

c) $\sum_{k=j}^{i-1} \lambda^k \leq \frac{r_i - r_j}{w}$ for all $i \in \{j+1, j+2, \dots, c\}$.

Conditions a) – c) arise from (3) with λ instead of l and by separating all known variables, i.e., decision variable $\boldsymbol{r}$ and realization w, from random variable λ. These conditions ensure that purchasing j units yields at least the same utility for customers as purchasing nothing (condition a)), purchasing less than j units (condition b)), and purchasing more than j units (condition c)).

Based on these conditions, there are several ways to eliminate demand for j units:

- Picking batch price $r_j > j \cdot w$ makes it impossible to fulfill condition a) for any $\lambda \in [0,1]$.
- If $r_j > r_i$ for any $i > j$, then there is no $\lambda \in [0,1]$ that satisfies condition c).
- Picking r_j such that $\left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}} > 1$ makes it impossible to fulfill condition b) for any $\lambda \in [0,1]$.
- With batch prices r_{j-1}, r_j, and r_{j+1} such that $\left(\frac{r_{j+1} - r_j}{w}\right)^{\frac{1}{j}} < \left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}}$, there is no $\lambda \in [0,1]$ such that $\lambda^{j-1} \geq \frac{r_j - r_{j-1}}{w}$ and $\lambda^j \leq \frac{r_{j+1} - r_j}{w}$ simultaneously (conditions b) and c) with $i = j - 1$ and $i = j + 1$, respectively).

In a scenario characterized by limited capacity, it becomes crucial to possess the capability to eliminate demand for any batch size j. There are two primary reasons why we seek this capability: firstly, we might encounter a situation where our capacity c is insufficient to fulfill an order of j units (i.e., $c < j$), and secondly, it may be more financially advantageous to reserve capacity for potential future customers. The latter circumstance arises when we are currently serving a customer with an exceptionally low willingness-to-pay, which is indicated by an exceedingly low value of w. We can establish a formal criterion for w being too low by referring to equation (7). This equation reveals that r_j should exceed $\Delta_j V_{t-1}(c)$ to increase overall expected revenue. The maximum possible willingness-to-pay for j units by a customer is given by $j \cdot w$ (as per equation (1) with $\lambda = 1$). When dealing with a customer whose w falls below $\frac{\Delta_j V_{t-1}(c)}{j}$, there is no viable way to sell j units without incurring a loss in overall expected revenue. In such cases, the firm's preference is not to sell j units to this customer, and we must ensure that at least one $\boldsymbol{r}$ is feasible such that $p_j(\boldsymbol{r}) = 0$.

Referring back to the previous points, we have ascertained that there exist numerous potential choices of r_j to eliminate demand for j units. Given that the primary goal of these r_j is to abstain from selling, it becomes immaterial which specific r_j is employed for this purpose. These observations prompt us to exclude the majority, though not all, of these alternatives from the action space $\mathcal{R}_c$. In the ensuing lemma, we define a refined action space that assumes a crucial role in this section. This set is denoted as $\mathcal{R}_c(w)$ and its elements are referred to as relevant prices, as we have removed only those prices deemed irrelevant.

Lemma 1 *Relevant prices* $\boldsymbol{r}$ *are given by*

$$\mathcal{R}_c(w) = \left\{ \boldsymbol{r} \in \mathbb{R}^c : 0 \leq r_1 \leq w^+, \text{and } 0 \leq \left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}} \leq \left(\frac{r_{j+1} - r_j}{w}\right)^{\frac{1}{j}} \leq 1 \ \text{ for } 2 \leq j \leq c-1 \right\}.$$

Proof: Firstly, it is essential to recognize that the definition of $\mathcal{R}_c(w)$ is derived exclusively by excluding any price vector that satisfies one of the conditions outlined in the bullet points above. Specifically, the first and third bullet points correspond to $r_1 \leq w^+$ and $\left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}} \leq 1$, the second to $0 \leq r_1$ and $0 \leq \left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}}$, and the fourth to $\left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}} \leq \left(\frac{r_{j+1} - r_j}{w}\right)^{\frac{1}{j}}$.

The fundamental concept behind this proof is straightforward: We show that for any excluded price vector, there exists a price vector $\boldsymbol{r} \in \mathcal{R}_c(w)$ that results in the same customer decisions and earned revenues. W.l.o.g., let us assume that an excluded price vector satisfies any of the bullet points for some j (if there are multiple instances, we iteratively apply the following steps). The implication is that demand for j units is eliminated. By substituting a certain value for r_j, we can ensure that demand for j units is still eliminated, while the resulting price vector belongs to $\mathcal{R}_c(w)$.

Considering the bullet points mentioned earlier, we want to shortly discuss what happens if we replace the inequality of these conditions with equality: Thereby, there is at most one $\lambda \in [0,1]$ such that conditions a) to c) are fulfilled. As we assume λ to be a continuously distributed random variable, the probability of λ being exactly this value is zero. Thus, we can eliminate demand almost surely by choosing r_j such that $r_j = j \cdot w$ (first bullet point with "="), $r_j = r_i$ (second bullet point with "="), $\left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}} = 1$ (third bullet point with "="), or $\left(\frac{r_{j+1} - r_j}{w}\right)^{\frac{1}{j}} = \left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}}$ (fourth bullet point with "="). Please note that the definition of $\mathcal{R}_c(w)$ always covers at least one of these four alternatives. This is sufficient for the purpose of maximizing expected revenue, and we can exclude all the cases mentioned in the bullet points without limiting possibilities for our optimization problem. □

In equation (1), we can observe that $\left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}}$ represents the lowest realization $l \in [0,1]$, for which a customer has nonnegative marginal utility when purchasing the jth unit ($j \geq 2$): $u_j(\boldsymbol{r}) - u_{j-1}(\boldsymbol{r}) = (X_j - r_j) - (X_{j-1} - r_{j-1}) = (X_j - X_{j-1}) - (r_j - r_{j-1}) = w \cdot \lambda^{j-1} - (r_j - r_{j-1}) \geq 0 \Leftrightarrow \lambda \geq \left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}}$. This threshold is crucial, and we define

$$\underline{l}_j(r_j - r_{j-1}) = \left(\frac{r_j - r_{j-1}}{w}\right)^{\frac{1}{j-1}} \qquad \text{for } j = 2, \dots, c \qquad (8)$$

Let us discuss the implication of this threshold in a short example: Assume we are dealing with a customer with a specific observable base willingness-to-pay (e.g., $w = 0.8$) and an unobservable consumption indicator λ with realizations $l \in [0, 1]$. The firm quotes an arbitrary price vector $\boldsymbol{r}$ with $\boldsymbol{r} \in \mathcal{R}_c(w)$ (e.g., the same price vector as depicted in Figure 2). Now, we can calculate for every batch size j the marginal utility $u_j(\boldsymbol{r}) - u_{j-1}(\boldsymbol{r})$. As the marginal utility depends on random variable λ, we portray it as function of every possible realization $l \in [0, 1]$, i.e., $l \longmapsto w \cdot l^{j-1} - (r_j - r_{j-1})$. To provide a clear illustration in Figure 3, we only show the marginal utilities for $j \leq 3$.

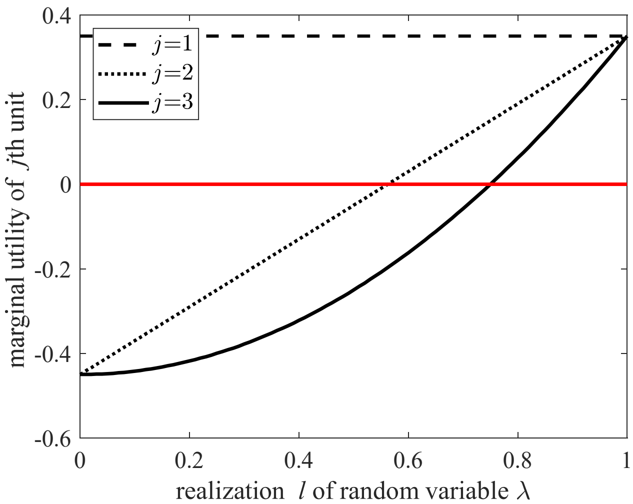

Figure 3: Exemplary marginal utility for $j \leq 3$ depending on realization l

The marginal utility for the first unit is positive for all $l \in [0, 1]$. Therefore, in this example, every customer with $w = 0.8$ prefers purchasing one unit over purchasing nothing at all. The marginal utility for the second and third unit becomes positive at $\underline{l}_2(r_2 - r_1)$ and $\underline{l}_3(r_3 - r_2)$, respectively. Customers with $l \geq \underline{l}_2(r_2 - r_1)$ and $l \geq \underline{l}_3(r_3 - r_2)$ can increase their utility by purchasing the second and third unit, respectively. We can now partition the interval $[0, 1]$ into $[0, \underline{l}_2(r_2 - r_1)]$, $[\underline{l}_2(r_2 - r_1), \underline{l}_3(r_3 - r_2)]$, and $[\underline{l}_3(r_3 - r_2), 1]$. Customers with $l = \underline{l}_j(r_j - r_{j-1})$ almost surely do not arrive (remember, λ is continuously distributed). Thus, it is irrelevant which of the adjacent intervals contains them. For presentation purposes, we decided to include them in both and work with closed intervals. Customers belonging to the first interval (based on their personal l) have positive marginal utility for purchasing one unit. They also have negative marginal utility for purchasing the second and third unit. Consequently, these customers attain their maximal utility by purchasing one unit. Analogously, customers belonging to the second and third interval decide to purchase two and three units, respectively.

We can generalize these considerations and partition $[0, 1]$ into $[0, \underline{l}_2(r_2 - r_1)]$, $[\underline{l}_j(r_j - r_{j-1}), \underline{l}_{j+1}(r_{j+1} - r_j)]$ for $j = 2, 3, \dots, c - 1$, and $[\underline{l}_c(r_c - r_{c-1}), 1]$. For $\boldsymbol{r} \in \mathcal{R}_c(w)$, by definition,

$\underline{l}_j(r_j - r_{j-1})$ is increasing in j. Hence, these intervals are well-defined, cover the entire interval $[0, 1]$, and are ordered such that $[\underline{l}_j(r_j - r_{j-1}), \underline{l}_{j+1}(r_{j+1} - r_j)]$ contains lower values than $[\underline{l}_i(r_i - r_{i-1}), \underline{l}_{i+1}(r_{i+1} - r_i)]$ if $j < i$. Moreover, we can conclude that customers belonging to $[\underline{l}_j(r_j - r_{j-1}), \underline{l}_{j+1}(r_{j+1} - r_j)]$ decide to purchase j units (as they have positive marginal utilities for $i \leq j$ and negative marginal utilities for $i > j$). Building on this, we can easily calculate the probability a customer purchase j units ($1 < j < c$) by calculating the probability a customer belongs to $[\underline{l}_j(r_j - r_{j-1}), \underline{l}_{j+1}(r_{j+1} - r_j)]$:

$$\mathbb{P}\left(l \in [\underline{l}_j(r_j - r_{j-1}), \underline{l}_{j+1}(r_{j+1} - r_j)]\right) = F_\lambda\left(\underline{l}_{j+1}(r_{j+1} - r_j)\right) - F_\lambda\left(\underline{l}_j(r_j - r_{j-1})\right).$$

This remains true for $j = c$ by replacing $\underline{l}_{c+1}(r_{c+1} - r_c)$ with 1. However, the $j = 1$ case is somewhat distinct: As utility for purchasing the first unit is independent of l, the marginal utility is constant. With $r_1 \leq w^+$ (one of the conditions that defines $\mathcal{R}_c(w)$), it is either positive ($r_1 < w$) or zero ($r_1 = w$ and $r_1 = w^+$). In the latter case, customers are indifferent between purchasing and not purchasing. We emphasize with $r_1 = w$ and $r_1 = w^+$ which of these equally viable options customers choose. As w and w^+ are the same value, the resulting intervals, i.e., $[0, \underline{l}_2(r_2 - r_1)]$, are identical in the sense that they cover the same area. And yet, they differ in meaning. One represents all customers that purchase exactly one unit (resulting from $r_1 = w$), the other represents all customers that purchase nothing at all (resulting from $r_1 = w^+$). Please note that this ambiguity has no impact on customers that belong to $[\underline{l}_j(r_j - r_{j-1}), \underline{l}_{j+1}(r_{j+1} - r_j)]$ with $j \geq 2$. These customers have a zero valued marginal utility for the first unit, a positive marginal utility for each ith unit with $i \leq j$, and negative marginal utilities for every other unit. Thus, they still attain their maximal utility by purchasing j units.

These considerations were enabled by restricting the action space to $\mathcal{R}_c(w)$. Only with this restriction, the intervals are guaranteed to be correctly ordered which, in turn, allows us to simplify the formulation of the selling probability $p_j(\boldsymbol{r}|w)$ for a batch of size j:

$$p_j(\boldsymbol{r}|w) = \begin{cases} F_\lambda\left(\underline{l}_{j+1}(r_{j+1} - r_j)\right) - F_\lambda\left(\underline{l}_j(r_j - r_{j-1})\right) & \text{for } 1 \leq j \leq c-1 \\ 1 - F_\lambda\left(\underline{l}_c(r_c - r_{c-1})\right) & \text{for } j = c \end{cases} \tag{9}$$

with $\underline{l}_1(r_1 - r_0) = \underline{l}_2(r_2 - r_1) \cdot 1_{\{r_1 = w^+\}}$ to properly reflect the ambiguous behavior of customers belonging to $[0, \underline{l}_2(r_2 - r_1)]$.

The optimization problem with observable base willingness-to-pay is given by

$$V_t^\omega(c) = \int_0^1 \max_{\boldsymbol{r} \in \mathcal{R}_c(w)} \left\{\sum_{j=1}^c p_j(\boldsymbol{r}|w) \cdot \left(r_j - \Delta_j V_{t-1}^\omega(c)\right)\right\} f_\omega(w)\, dw + V_{t-1}^\omega(c) \tag{10}$$

with boundary conditions $V_0^{\omega}(c) = 0$ for $c \geq 0$ and $V_t^{\omega}(0) = 0$ for $t \geq 0$. By definition of $p_j(\boldsymbol{r}|w)$, r_j influences the probability of three possible outcomes: selling $j-1$, j, and $j+1$ units. This interconnection thwarts maximizing $\sum_{j=1}^{c} p_j(\boldsymbol{r}|w) \cdot \left(r_j - \Delta_j V_{t-1}^{\omega}(c)\right)$ separately. We can circumvent this obstacle by defining $\Delta r_j = r_j - r_{j-1}$ for $j \leq c$ $(r_0 = 0)$ and reformulating $\sum_{j=1}^{c} p_j(\boldsymbol{r}|w) \cdot \left(r_j - \Delta_j V_{t-1}^{\omega}(c)\right) = \sum_{j=1}^{c} p_j(\boldsymbol{r}|w) \cdot \left(\sum_{i=1}^{j} \Delta r_i - \Delta_1 V_{t-1}^{\omega}(c+1-i)\right) = \sum_{j=1}^{c} \left(1 - F_\lambda\left(\underline{l}_j(\Delta r_j)\right)\right) \cdot \left(\Delta r_j - \Delta_1 V_{t-1}^{\omega}(c+1-j)\right)$ for $\boldsymbol{r} \in \mathcal{R}_c(w)$. Moreover, with (8), we write $\mathcal{R}_c(w) = \{\boldsymbol{r} \in \mathbb{R}^c : 0 \leq \Delta r_1 \leq w^+$, and $0 \leq \underline{l}_j(\Delta r_j) \leq \underline{l}_{j+1}(\Delta r_{j+1}) \leq 1$ for $2 \leq j \leq c-1\}$. The only remaining connection between marginal unit prices Δr_j is given by the imposed order $\underline{l}_j(\Delta r_j) \leq \underline{l}_{j+1}(\Delta r_{j+1})$ for $2 \leq j \leq c-1$. When defining $\underline{l}_j(\Delta r_j)$ in (8), we showed that this is the threshold between negative ($l < \underline{l}_j(\Delta r_j)$) and positive ($l > \underline{l}_j(\Delta r_j)$) marginal utility for purchasing the jth unit. So, in conclusion, this order ensures a pricing scheme where customers only consider buying the $j+1$th unit if they also buy the jth.

Currently, this order, imposed by conditions $\underline{l}_j(\Delta r_j) \leq \underline{l}_{j+1}(\Delta r_{j+1})$, $2 \leq j \leq c-1$, is preventing us from optimizing every decision variable independently. By removing these conditions, we formulate an optimization problem that is entirely separable in each decision variable and serves as an upper bound to (10):

$$\sum_{j=1}^{c} \max_{\Delta r_j \in [0,w]} \left\{\left(1 - F_\lambda\left(\underline{l}_j(\Delta r_j)\right)\right) \cdot \left(\Delta r_j - \Delta_1 V_{t-1}^{\omega}(c+1-j)\right)\right\}. \tag{11}$$

The roadmap for the remaining section is as follows: We first determine the solution of upper bound problem (11), show that under certain conditions this solution is also the solution of (10) resulting in the same expected revenue, and finally show by induction that these conditions are indeed met.

4.1.2. Solution and structural properties

For every j, we check if we can economically sell the jth unit. We use the term "economic selling" to refer to selling an additional unit at a price that covers at least the lost expected revenue of the additionally sold capacity (opportunity cost), i.e. $\Delta r_j \geq \Delta_1 V_{t-1}^{\omega}(c-j+1)$. Whenever $\Delta_1 V_{t-1}^{\omega}(c+1-j)$ is exceeding w, we cannot economically sell the jth unit and choose to eliminate demand for it, i.e. we pick $\Delta r_j = w$.

Corollary 1 *If $V_{t-1}^{\omega}(\cdot)$ is increasing and concave, $N_{t,c}(w) = \max_{j=1,\dots,c} \{j : \Delta_1 V_{t-1}^{\omega}(c-j+1) < w\}$ denotes the highest additional unit that can be sold economically. It holds that $\{j : \Delta_1 V_{t-1}^{\omega}(c-j+1) < w\} = \{1, 2, \dots, N_{t,c}(w)\}$.*

Proof: As $V_{t-1}^{\omega}(\cdot)$ is concave, $\Delta_1 V_{t-1}^{\omega}(c-j+1)$ is increasing in j. Thus, there is $\hat{j} \in \{1,2,\dots,c\}$ with $\Delta_1 V_{t-1}^{\omega}(c-j+1) < w \Leftrightarrow j \leq \hat{j}$. □

By definition, it holds that $\Delta_1 V_{t-1}^{\omega}\big(c - N_{t,c}(w) + 1\big) < w$ and $\Delta_1 V_{t-1}^{\omega}\big(c - \big(N_{t,c}(w) + 1\big) + 1\big) \geq w$. Consequently, it also holds that $\Delta_1 V_{t-1}^{\omega}\Big((c-1) - \big(N_{t,c}(w) - 1\big) - 1\Big) < w$ and $\Delta_1 V_{t-1}^{\omega}\Big((c-1) - N_{t,c}(w) + 1\Big) \geq w$ as well as $\Delta_1 V_{t-1}^{\omega}\Big((c+1) - \big(N_{t,c}(w) + 1\big) - 1\Big) < w$ and $\Delta_1 V_{t-1}^{\omega}\Big((c+1) - \big(N_{t,c}(w) + 2\big) + 1\Big) \geq w$. This observation leads to the following remark.

Remark 1 *It holds that* $N_{t,c-1}(w) + 1 = N_{t,c}(w) = N_{t,c+1}(w) - 1$.

The solution to $\max_{\Delta r_1 \in [0,w]} \Big\{\Big(1 - F_\lambda\big(\underline{l}_1(\Delta r_1)\big)\Big) \cdot \big(\Delta r_1 - \Delta_1 V_{t-1}^{\omega}(c)\big)\Big\}$ is $\Delta r_1 = w$ (w^+ if $\Delta_1 V_{t-1}^{\omega}(c) \geq w$). However, for every other j, the solution is less apparent.

In the proof of Proposition 1, we show that there exists exactly one solution Δr_j to (11). We determine this solution with the help of the optimal customer threshold $\underline{l}_j$, which is (implicitly) defined in Proposition 1. There, we observe that the optimal solution depends on realization w and opportunity costs $\Delta_1 V_{t-1}^{\omega}(c + 1 - j)$.

Proposition 1 *In every state* (t, c) *and for every* $w \in [0,1]$, *there is a unique optimal solution* $\Delta r_{t,j}(c|w)$, $j \leq c$, *for* (11)*:*

- *If* $\Delta_1 V_{t-1}^{\omega}(c + 1 - j) \geq w$, $\Delta r_{t,j}(c|w) = w^+$ *with* $\underline{l}_j(w) = 1$.
- *If* $\Delta_1 V_{t-1}^{\omega}(c + 1 - j) \in [0, w)$, $\Delta r_{t,1}(c|w) = w$ *with* $\underline{l}_1(w) = 0$ *for* $j = 1$ *and* $\Delta r_{t,j}(c|w) = w \cdot \big(\underline{l}_j(w)\big)^{j-1}$ *with* $\underline{l}_j(w)$ *implicitly defined by* $w \cdot \underline{l}_j^{\,j-2}(w) \cdot \left(\underline{l}_j(w) - \frac{j-1}{h_\lambda\left(\underline{l}_j(w)\right)}\right) = \Delta_1 V_{t-1}^{\omega}(c + 1 - j)$ *for* $j \geq 2$.

Proof: We have already established that $\Delta r_{\mathrm{t},1}(c|w) = w$ when $\Delta_1 V_{t-1}^{\omega}(c + 1 - j) \in (0, w)$. Hence, our focus will be on $j \geq 2$ in the subsequent discussion.

In this proof, we aim to achieve two objectives. First, we intend to derive the implicit definition and argue that there is at least one solution meeting this criterion. Second, we aim to prove that there could only be one solution meeting this criterion. To accomplish the first goal, we will formulate the first-order condition and examine the values of the first derivative at the interval boundaries. The second goal will be secured by establishing the second derivative and demonstrating its negativity for every solution that satisfies the first-order condition. With the continuity of the first derivative, this is sufficient to conclude that there is exactly one point where the first derivative equals zero. Hence, there is exactly one solution meeting the first-order condition and maximizing the optimization problem.

We commence with a reformulation of the optimization problem, a convenient step to simplify the second derivative.

There are different approaches to tackle this optimization problem: we can try to find the optimal marginal price Δr_j, the optimal customer threshold $\underline{l}_j$, or the optimal probability $\theta = 1 - F_\lambda\big(\underline{l}_j\big)$. As

$\underline{l}_j(\Delta r_j) = \left(\frac{\Delta r_j}{w}\right)^{\frac{1}{j-1}}$ is bijective on $[0, w]$, and the distribution function is bijective on its support $[0, 1]$, there is a unique mapping between Δr_j, $\underline{l}_j$, and θ. This enables us to treat each of these variables as a decision variable and use the mapping to calculate the other two.

In this proof, it is more convenient to focus on θ as our decision variable. Thereby, we do not have to deal with (varying) opportunity costs in the second derivative. We reformulate our optimization problem with $\Delta r_j = w \cdot (\underline{l}_j)^{j-1}$, $\theta = 1 - F_\lambda(\underline{l}_j)$, and F_λ^{-1} being the inverse to F_λ:

$$\max_{\Delta r_j \in [0,w]} \left\{ \left(1 - F_\lambda\left(\left(\frac{\Delta r_j}{w} \right)^{\frac{1}{j-1}} \right) \right) \cdot \left(\Delta r_j - \Delta_1 V_{t-1}^{\omega}(c+1-j) \right) \right\}$$

$$= \max_{\underline{l}_j \in [0,1]} \left\{ \left(1 - F_\lambda(\underline{l}_j)\right) \cdot \left(w \cdot (\underline{l}_j)^{j-1} - \Delta_1 V_{t-1}^{\omega}(c+1-j) \right) \right\}$$

$$= \max_{\theta \in [0,1]} \left\{ \theta \cdot \left(w \cdot \left(F_\lambda^{-1}(1-\theta)\right)^{j-1} - \Delta_1 V_{t-1}^{\omega}(c+1-j) \right) \right\}.$$

We can now approach our first goal. The optimal solution has to meet the first-order condition:

$$\frac{d}{d\theta} \theta \cdot \left(w \cdot \left(F_\lambda^{-1}(1-\theta)\right)^{j-1} - \Delta_1 V_{t-1}^{\omega}(c+1-j) \right)$$

$$= w \cdot \left(F_\lambda^{-1}(1-\theta)\right)^{j-1} - \Delta_1 V_{t-1}^{\omega}(c+1-j) - \theta \cdot w \cdot (j-1) \cdot \left(F_\lambda^{-1}(1-\theta)\right)^{j-2}$$

$$\cdot \frac{1}{f_\lambda\left(F_\lambda^{-1}(1-\theta)\right)} = 0.$$

This condition is well-defined as $f_\lambda > 0$ on the distribution's support $[0, 1]$. The existence of a solution is ensured by the continuity of the first derivative as well as the fact that it is non-negative for $\theta = 1$, and positive for $\theta = 0$ (remember that $w > \Delta_1 V_{t-1}^{\omega}(c+1-j)$).

With $\underline{l}_j = F_\lambda^{-1}(1-\theta)$ and the definition of the failure rate, we can reformulate the first-order condition to

$$w \cdot (\underline{l}_j)^{j-1} - \Delta_1 V_{t-1}^{\omega}(c+1-j) - w \cdot (j-1) \cdot (\underline{l}_j)^{j-2} \cdot \frac{1}{h_\lambda(\underline{l}_j)} = 0.$$

To achieve our second objective, we derive the second derivative on this formulation and write $\underline{l}_j(\theta)$ to emphasize that $\underline{l}_j$ depends on θ (our decision variable in this proof). Subsequently, we demonstrate that the second derivative is negative for every θ that satisfies the first-order condition. With the continuity of the first derivative, this is sufficient to establish the uniqueness of such a θ:

$$\frac{d^2}{d\theta^2}\theta \cdot \left(w \cdot \left(F_\lambda^{-1}(1-\theta)\right)^{j-1} - \Delta_1 V_{t-1}^{\omega}(c+1-j)\right)$$

$$= \frac{d}{d\theta} w \cdot \left(\underline{l}_j(\theta)\right)^{j-1} - \Delta_1 V_{t-1}^{\omega}(c+1-j) - w \cdot (j-1) \cdot \left(\underline{l}_j(\theta)\right)^{j-2} \cdot \frac{1}{h_\lambda\left(\underline{l}_j(\theta)\right)}$$

$$= w \cdot (j-1) \cdot \left(\underline{l}_j(\theta)\right)^{j-3} \left(\underline{l}_j(\theta) - \frac{(j-2) \cdot h_\lambda\left(\underline{l}_j(\theta)\right) - \underline{l}_j(\theta) \cdot h_\lambda'\left(\underline{l}_j(\theta)\right)}{\left(h_\lambda\left(\underline{l}_j(\theta)\right)\right)^2}\right)$$

$$\cdot \frac{d}{d\theta}\underline{l}_j(\theta)$$

for $j \geq 3$, and

$$\frac{d^2}{d\theta^2}\theta \cdot \left(w \cdot \left(F_\lambda^{-1}(1-\theta)\right)^{j-1} - \Delta_1 V_{t-1}^{\omega}(c+1-j)\right)$$

$$= \left(w + w \cdot (j-1) \cdot \frac{h_\lambda'\left(\underline{l}_j(\theta)\right)}{\left(h_\lambda\left(\underline{l}_j(\theta)\right)\right)^2}\right) \cdot \frac{d}{d\theta}\underline{l}_j(\theta) < 0$$

for $j = 2$. With $\frac{d}{d\theta}\underline{l}_j(\theta) < 0$ and $h_\lambda'\left(\underline{l}_j(\theta)\right) \geq 0$, the latter case is trivial. Thus, we will focus on $j \geq 3$ for the remaining part of the proof. Again with $\frac{d}{d\theta}\underline{l}_j(\theta) < 0$, we only need to show that $\underline{l}_j(\theta) - \frac{(j-2) \cdot h_\lambda\left(\underline{l}_j(\theta)\right) - \underline{l}_j(\theta) \cdot h_\lambda'\left(\underline{l}_j(\theta)\right)}{\left(h_\lambda\left(\underline{l}_j(\theta)\right)\right)^2} > 0$ for every θ that meets the first-order condition. It holds that

$$\underline{l}_j(\theta) - \frac{(j-2) \cdot h_\lambda\left(\underline{l}_j(\theta)\right) - \underline{l}_j(\theta) \cdot h_\lambda'\left(\underline{l}_j(\theta)\right)}{\left(h_\lambda\left(\underline{l}_j(\theta)\right)\right)^2}$$

$$= \underbrace{\Delta_1 V_{t-1}^{\omega}(c+1-j)}_{\geq 0} + \underbrace{\frac{1}{h_\lambda\left(\underline{l}_j(\theta)\right)}}_{>0} + \underbrace{\frac{\underline{l}_j(\theta) \cdot h_\lambda'\left(\underline{l}_j(\theta)\right)}{\left(h_\lambda\left(\underline{l}_j(\theta)\right)\right)^2}}_{\geq 0} > 0.$$

The equality follows by the first-order condition. Also note that h_λ is positive on the distribution's support and increasing by assumption. □

Remark 2 *For $\lambda \sim U[0,1]$ and $\Delta_1 V_{t-1}^{\omega}(c+1-j) = 0$, the optimality condition leads to a closed-form solution: $\Delta r_{t,j}(c|w) = w \cdot \left(\frac{j-1}{j}\right)^{j-1}, j \geq 2$.*

For now, we have (implicitly) given the solution of upper bound (11). If we can show that this solution is a feasible solution to (10), i.e. $\Delta r_{t,j}(c|w) \in \mathcal{R}_c(w)$, we can immediately conclude that $\Delta r_{t,j}(c|w)$ results in the same expected revenue in (10) and is the unique optimal solution. It holds that $\Delta r_{t,j}(c|w) \in \mathcal{R}_c(w) \Leftrightarrow \left(\Delta r_{t,1}(c|w) \in [0, w^+] \text{ and } \underline{l}_j(w) \leq \underline{l}_{j+1}(w) \text{ for } 2 \leq j \leq c-1\right)$.

In proof of Proposition 1, we have seen that $\underline{l}_j(w) < 1$ (resulting from $\theta > 0$) if $\Delta_1 V_{t-1}^{\omega}(c+1-j) < w$ and $\underline{l}_j(w) = 1$ if $\Delta_1 V_{t-1}^{\omega}(c+1-j) \geq w$. This could lead to contradicting condition $\underline{l}_j(w) \leq \underline{l}_{j+1}(w)$ if there is j such that $\Delta_1 V_{t-1}^{\omega}(c+1-j) \geq w$ and $\Delta_1 V_{t-1}^{\omega}(c-j) < w$. Therefore, a necessary condition for $\Delta r_{t,j}(c|w) \in \mathcal{R}_c(w)$ is $\Delta_1 V_{t-1}^{\omega}(c+1-\hat{\jmath}) \geq w \Rightarrow (\Delta_1 V_{t-1}^{\omega}(c+1-j) \geq w \quad \forall j \geq \hat{\jmath})$. This is ensured when $V_{t-1}^{\omega}(\cdot)$ is concave, so we will stick with this condition.

Proposition 2 *If $V_{t-1}^{\omega}(\cdot)$ is increasing and concave, $\Delta r_{t,j}(c|w)$ defined by Proposition 1 is the optimal solution for* (10).

Proof: We formulated optimization problem (11) by removing conditions $\underline{l}_j(w) \leq \underline{l}_{j+1}(w)$ for $2 \leq j \leq c-1$. Therefore, demonstrating that $\Delta r_{t,j}(c|w)$ satisfies these conditions is sufficient to show Proposition 2.
According to Corollary 1, $\Delta_1 V_{t-1}^{\omega}(c-j+1) < w$ holds for $j \leq N_{t,c}(w)$, and $\Delta_1 V_{t-1}^{\omega}(c-j+1) \geq w$ holds for $j > N_{t,c}(w)$. Combined with Proposition 1, this implies that $\underline{l}_j(w) = 1$ for $j > N_{t,c}(w)$, which evidently aligns with $\underline{l}_j(w) \leq \underline{l}_{j+1}(w)$. Therefore, in the subsequent discussion, we exclusively focus on the case where $j \leq N_{t,c}(w)$.
We know from Proposition 1 (and its proof) that $0 = \underline{l}_1(w) \leq \underline{l}_2(w)$, and $\underline{l}_j(w)$ such that $w \cdot \underline{l}_j^{j-2}(w) \cdot \left(\underline{l}_j(w) - \frac{j-1}{h_\lambda\left(\underline{l}_j(w)\right)}\right) = \Delta_1 V_{t-1}^{\omega}(c+1-j), j \geq 2$.
Focusing on $w \cdot \underline{l}^{j-2} \cdot \left(\underline{l} - \frac{j-1}{h_\lambda(\underline{l})}\right)$, we can observe that this formulation is decreasing in j if $\underline{l} - \frac{j-1}{h_\lambda(\underline{l})} \geq 0$. As $\underline{l}_j(w) - \frac{j-1}{h_\lambda\left(\underline{l}_j(w)\right)} \geq 0$, it holds that

$$\begin{aligned} 0 &= w \cdot \underline{l}_j^{j-2}(w) \cdot \left(\underline{l}_j(w) - \frac{j-1}{h_\lambda\left(\underline{l}_j(w)\right)}\right) - \Delta_1 V_{t-1}^{\omega}(c+1-j) \\ &> w \cdot \underline{l}_j^{j-1}(w) \cdot \left(\underline{l}_j(w) - \frac{j}{h_\lambda\left(\underline{l}_j(w)\right)}\right) - \Delta_1 V_{t-1}^{\omega}(c+1-j) \\ &\geq w \cdot \underline{l}_j^{j-1}(w) \cdot \left(\underline{l}_j(w) - \frac{j}{h_\lambda\left(\underline{l}_j(w)\right)}\right) - \Delta_1 V_{t-1}^{\omega}(c-j), \end{aligned}$$

$j \geq 2$, where the equation follows by Proposition 1, the first inequality by increasing j to $j+1$, and the last inequality by concavity of $V_{t-1}^{\omega}(\cdot)$. The negativity of the last term proves that the optimal solution $\underline{l}_j(w)$ for selling the jth unit does not satisfy the optimality condition for selling the $j+1$th unit. More precisely, it proves that the point where this optimality condition is fulfilled, denoted as $\underline{l}_{j+1}(w)$, must be positioned above $\underline{l}_j(w)$. In simpler terms, $\underline{l}_j(w) < \underline{l}_{j+1}(w)$. □

So far, we have shown $\Delta r_{t,j}(c|w)$ (implicitly) defined by Proposition 1 is the optimal solution to (10) for every t if $V_{t-1}^{\omega}(\cdot)$ is increasing and concave. We will show that this condition indeed holds for the whole horizon. In the upcoming proof, the optimal expected margin m_j for selling the jth unit, and its sensitivity to changes in opportunity costs, will play a crucial role. Hence, we introduce $m_j(\delta) = \max_{\Delta r_j \in [0,w]} \left\{ \left(1 - F_\lambda\left(\underline{l}_j(\Delta r_j)\right)\right) \cdot (\Delta r_j - \delta) \right\}$ as a function of variable δ which represents any opportunity costs. This allows us to analyze the impact of varying opportunity costs on the optimal expected margin. Lemma 2 outlines certain properties of $m_j(\delta)$ that will prove useful in establishing concavity of $V_{t-1}^{\omega}(\cdot)$ later in this section.

Lemma 2 *If $\delta \in [0, w)$, it holds that:*

a) $m_{j+1}(\delta) - m_j(\delta) \leq 0$

b) $m_{j+1}(\delta) - m_j(\delta)$ *is increasing in* δ

c) $m_j(\delta)$ *is decreasing in* δ

Proof: We will address a), b), and c) separately, though not in this order. To streamline the proof of b), we will employ a formulation derived in c), so we will modify the order accordingly.

a): The assertion that the optimal expected margin declines with j, i.e., $m_j(\delta) \geq m_{j+1}(\delta)$, is rooted in two observations: the suboptimality of the solution of $m_{j+1}(\delta)$ for $m_j(\delta)$, and the fact that expected margin decreases with j for any $l \in [0,1]$.

$m_j(\delta)$ is the optimal value of $\max_{\Delta r_j \in [0,w]} \left\{ \left(1 - F_\lambda\left(\underline{l}_j(\Delta r_j)\right)\right) \cdot (\Delta r_j - \delta) \right\} = \max_{\underline{l}_j \in [0,1]} \left\{ \left(1 - F_\lambda(\underline{l}_j)\right) \cdot \left(w \cdot (\underline{l}_j)^{j-1} - \delta\right) \right\} = \left(1 - F_\lambda\left(\underline{l}_j(w)\right)\right) \cdot \left(w \cdot \left(\underline{l}_j(w)\right)^{j-1} - \delta\right)$ with $\underline{l}_j(w)$ representing the optimal solution. As $\underline{l}_{j+1}(w)$ (the optimal solution of $m_{j+1}(\delta)$) is suboptimal for $m_j(\delta)$ and $1 - F_\lambda\left(\underline{l}_{j+1}(w)\right) \geq 0$ as well as $\underline{l}_{j+1}(w) \leq 1$, it holds that

$$m_j(\delta) = \max_{\underline{l}_j \in [0,1]} \left\{ \left(1 - F_\lambda(\underline{l}_j)\right) \cdot \left(w \cdot (\underline{l}_j)^{j-1} - \delta\right) \right\} \geq \left(1 - F_\lambda\left(\underline{l}_{j+1}(w)\right)\right) \cdot \left(w \cdot \left(\underline{l}_{j+1}(w)\right)^{j-1} - \delta\right)$$
$$\geq \left(1 - F_\lambda\left(\underline{l}_{j+1}(w)\right)\right) \cdot \left(w \cdot \left(\underline{l}_{j+1}(w)\right)^{j} - \delta\right) = m_{j+1}(\delta)$$

c): To prove c), we will derive the first derivative of $m_j(\delta)$ with respect to δ and demonstrate its nonpositivity.

Based on its implicit definition $w \cdot \underline{l}_j^{j-2}(w) \cdot \left(\underline{l}_j(w) - \frac{j-1}{h_\lambda\left(\underline{l}_j(w)\right)}\right) = \delta$ (refer to Proposition 1), the optimal solution $\underline{l}_j(w)$ of $m_j(\delta)$ depends also on δ. As we are about to vary δ, we highlight this fact by writing $\underline{l}_j(\delta)$ instead of $\underline{l}_j(w)$ (w acts as a parameter in this proof). The same applies for $\underline{l}_{j+1}(\delta)$ and $m_{j+1}(\delta)$. Building the first derivative, we get

$$\frac{d}{d\delta}m_j(\delta) = \frac{d}{d\delta}\left(\left(1 - F_\lambda\left(\underline{l}_j(\delta)\right)\right) \cdot \left(w \cdot \left(\underline{l}_j(\delta)\right)^{j-1} - \delta\right)\right)$$

$$= -f_\lambda\left(\underline{l}_j(\delta)\right) \cdot \frac{d}{d\delta}\left(\underline{l}_j(\delta)\right) \cdot \left(w \cdot \left(\underline{l}_j(\delta)\right)^{j-1} - \delta\right) + \left(1 - F_\lambda\left(\underline{l}_j(\delta)\right)\right)$$

$$\cdot \left(w \cdot (j-1) \cdot \left(\underline{l}_j(\delta)\right)^{j-2} \cdot \frac{d}{d\delta}\left(\underline{l}_j(\delta)\right) - 1\right)$$

$$= \frac{d}{d\delta}\left(\underline{l}_j(\delta)\right) \cdot f_\lambda\left(\underline{l}_j(\delta)\right) \cdot \left(w \cdot \left(\underline{l}_j(\delta)\right)^{j-2} \cdot \left(\frac{j-1}{h_\lambda\left(\underline{l}_j(\delta)\right)} - \underline{l}_j(\delta)\right) + \delta\right)$$

$$- \left(1 - F_\lambda\left(\underline{l}_j(\delta)\right)\right) = -\left(1 - F_\lambda\left(\underline{l}_j(\delta)\right)\right) \leq 0.$$

The last equation holds because of the implicit definition of $\underline{l}_j(\delta)$.

<u>b):</u> Similarly to c), we aim to calculate the first derivative $\frac{d}{d\delta}\left(m_{j+1}(\delta) - m_j(\delta)\right)$. Fortunately, we can leverage the first derivative of $m_j(\delta)$ with respect to δ. It is important to note that substituting j by $j+1$ does not alter the reasoning in c). Consequently, we find that $\frac{d}{d\delta}m_{j+1}(\delta) = -\left(1 - F_\lambda\left(\underline{l}_{j+1}(\delta)\right)\right)$. Combining the first derivative of $m_j(\delta)$ and $m_{j+1}(\delta)$ leads to

$$\frac{d}{d\delta}\left(m_{j+1}(\delta) - m_j(\delta)\right) = F_\lambda\left(\underline{l}_{j+1}(\delta)\right) - F_\lambda\left(\underline{l}_j(\delta)\right).$$

Recalling the argumentation while developing Proposition 2, we know that $\underline{l}_{j+1}(\delta) \geq \underline{l}_j(\delta)$. Hence, we can conclude that $\frac{d}{d\delta}\left(m_{j+1}(\delta) - m_j(\delta)\right) \geq 0$. □

Even though we developed Lemma 2 mainly to show the desired concavity of $V_t^\omega(\cdot)$, it also brings interesting implications with it: The optimal expected margin for selling the jth unit is greater than the optimal expected margin for selling the $j+1$th unit given both cases result in the same additional opportunity costs. With a concave value function $V_t^\omega(\cdot)$, we can conclude that selling the $j+1$th unit results in higher additional opportunity costs and, thus, selling the $j+1$th unit definitely leads to a lower optimal expected margin than selling the jth unit does.

Before delving into the proof of the preservation of concavity across periods, let us examine a small example. Assume that ω and λ follow a uniform distribution. We address the optimization problem for all states (t, c) with $t = 1, 2$ and $c = 1, \ldots, 5$. We start with $t = 1$, as $V_2^\omega(\cdot)$ depends on $V_1^\omega(\cdot)$ which in turn depends on $V_0^\omega(\cdot)$. After the selling horizon, no revenue can be earned, leading to the boundary condition $V_0^\omega(c) = 0$ for $c \geq 0$.

In $t = 1$, observe that $V_0^\omega(c)$ is (as a constant, not strictly) increasing and concave. Propositions 1 and 2 allow us to calculate $V_1^\omega(c)$. In addition, with no opportunity costs ($\Delta_1 V_0^\omega(c) = 0$), we can use the closed-form expression of the optimal solution from Remark 2, i.e., $\Delta r_1(c|w) = w$ and $\Delta r_j(c|w) =$

$w \cdot \left(\frac{j-1}{j}\right)^{j-1}$, for every possible realization w of ω. This yields a closed-form expression for $V_1^{\omega}(c|w) = V_0^{\omega}(c) + \sum_{j=1}^{c} m_j(0) = V_0^{\omega}(c) + w \cdot 1_{\{w > \Delta_1 V_0^{\omega}(c)\}} + \sum_{j=2}^{c} \left(1 - F_\lambda\left(\frac{j-1}{j}\right)\right) \cdot w \cdot \left(\frac{j-1}{j}\right)^{j-1}$ and we calculate $V_1^{\omega}(c) = \int_0^1 V_1^{\omega}(c|w) \cdot f_\omega(w)\, dw$. The results of these calculations are presented in Table 1, revealing that $V_1^{\omega}(c)$ is increasing in c. As $\Delta_1 V_1^{\omega}(c) = V_1^{\omega}(c) - V_1^{\omega}(c-1)$ is decreasing in c, $V_1^{\omega}(c)$ is also concave.

Moving to $t = 2$, Propositions 1 and 2 remain applicable (we just observed that $V_1^{\omega}(c)$ is increasing and concave). However, Remark 2 is no longer relevant ($\Delta_1 V_1^{\omega}(c) \neq 0$). Consequently, we can no longer rely on the closed-form expression of the optimal solution. Without the closed-form solution, solving the optimization problem for every realization w becomes more challenging. As an example, we focus on the specific realization $w = 0.1$ and $c = 5$ in detail.

The maximal number of units that can be sold economically is given by $N_{2,5}(0.1) = \max_{j=1,\ldots,5} \{j\colon \Delta_1 V_{t-1}^{\omega}(6-j) < 0.1\} = 3$. Consequently, the optimization problem becomes $V_2^{\omega}(5|0.1) = V_1^{\omega}(5) + \sum_{j=1}^{3} m_j\left(\Delta_1 V_1^{\omega}(6-j)\right) = V_1^{\omega}(5) + \left(0.1 - \Delta_1 V_1^{\omega}(5)\right) + m_2\left(\Delta_1 V_1^{\omega}(4)\right) + m_3\left(\Delta_1 V_1^{\omega}(3)\right)$.

To apply the optimality condition, we note that $\frac{1}{h_\lambda\left(\underline{l}_j(w)\right)} = \frac{1 - F_\lambda\left(\underline{l}_j(w)\right)}{f_\lambda\left(\underline{l}_j(w)\right)} = 1 - \underline{l}_j(w)$ for $\lambda \sim U[0,1]$. The optimality condition becomes $w \cdot \underline{l}_j^{\,j-2}(w) \cdot \left(j \cdot \underline{l}_j(w) - (j-1)\right) = \Delta_1 V_{t-1}^{\omega}(c+1-j)$.

Optimal Δr_2 is given by $\Delta r_2 = 0.1 \cdot \underline{l}_2(0.1)$ with $\underline{l}_2(0.1)$ such that $0.1 \cdot \left(2 \cdot \underline{l}_2(0.1) - 1\right) = \Delta_1 V_1^{\omega}(4)$. Thus, $\underline{l}_2(0.1) = \frac{(10 \cdot \Delta_1 V_1^{\omega}(4) + 1)}{2} \approx 0.7635$ and $\Delta r_2 = \left(\frac{\Delta_1 V_1^{\omega}(4) + 0.1}{2}\right) \approx 0.0764$. Optimal Δr_3 is given by $\Delta r_3 = 0.1 \cdot \left(\underline{l}_3(0.1)\right)^2$ with $\underline{l}_3(0.1)$ such that $0.1 \cdot \underline{l}_3(0.1) \cdot \left(3 \cdot \underline{l}_3(0.1) - 2\right) = \Delta_1 V_1^{\omega}(3)$. Thus, $\underline{l}_3(0.1) = \frac{2 + \sqrt{4 + 120 \cdot \Delta_1 V_1^{\omega}(3)}}{6} \approx 0.9318$ and $\Delta r_3 \approx 0.0868$. Consequently, $V_2^{\omega}(5|0.1) \approx 0.7928 + 0.059 + 0.0056 + 0.0009 \approx 0.8583$.

Table 1: Example with $\boldsymbol{\omega, \lambda \sim U[0,1]}$ and $\boldsymbol{c \leq 5, t = 1, 2}$

	$V_2^{\omega}(c)$	$V_2^{\omega}(c|0.1)$	$\Delta_1 V_1^{\omega}(c)$	$V_1^{\omega}(c)$
$\boldsymbol{c = 1}$	0.6250	0.5000	0.5000	0.5000
$\boldsymbol{c = 2}$	1.0199	0.6250	0.1250	0.6250
$\boldsymbol{c = 3}$	1.2106	0.7250	0.0741	0.6991
$\boldsymbol{c = 4}$	1.3419	0.8008	0.0527	0.7518
$\boldsymbol{c = 5}$	1.4420	0.8583	0.0410	0.7928

Similarly, we can calculate $N_{2,4}(0.1) = 2$, $N_{2,3}(0.1) = 1$, and $N_{2,2}(0.1) = N_{2,1}(0.1) = 0$ as well as $V_2^\omega(c|0.1)$, $c \leq 4$ (cf, Table 1). Once again, we observe that $V_2^\omega(c|0.1)$ increases, and $V_2^\omega(c|0.1) - V_2^\omega(c-1|0.1)$ decreases in c.

Finally, we numerically derive $V_2^\omega(c)$, $c \leq 5$, and, again, observe that these properties are still intact.

In our example, we have seen that these conditions stayed intact. Now, we want to prove that these conditions indeed hold for every $t \leq T$ and any distribution that meets the assumption formulated in Section 3.2.

Proposition 3 *For every t, $V_t^\omega(\cdot)$ is increasing and concave.*

Proof: See Supplement S.1.

Proposition 3 confirms the optimality of prices defined by Proposition 1 in a scenario where base willingness-to-pay is observable. The optimality condition is influenced by two factors: the specific customer type indicated by the observed base willingness-to-pay and opportunity costs. While the former is stochastic and, hence, ex ante unpredictable, the latter is state-dependent and can be determined beforehand. Consequently, understanding the dynamics of opportunity costs is crucial for comprehending the optimal pricing policy. The subsequent proposition illustrates how opportunity costs and the value function evolve over time. Notably, the increase in opportunity costs over time is intriguing, suggesting that optimal marginal prices may also experience an upward trend.

Proposition 4 *For every c, it holds:*

a) $\Delta_1 V_t^\omega(c)$ *is increasing in* t
b) $V_t^\omega(c)$ *is increasing and concave in* t

Proof: See Supplement S.2.

For a first impression regarding dynamics of optimal prices, we start with a generic look at the optimality condition given by Proposition 1. We will use δ as variable for opportunity costs, and replace $\underline{l}_j(w)$ with $\left(\frac{\Delta r_j}{w}\right)^{\frac{1}{j-1}}$. With some algebra, we can reformulate the (sufficient) first-order condition to

$$\frac{\delta}{(j-1)\cdot\Delta r_j} + \frac{1}{h_\lambda\left(\left(\frac{\Delta r_j}{w}\right)^{\frac{1}{j-1}}\right)\cdot\left(\frac{\Delta r_j}{w}\right)^{\frac{1}{j-1}}} = \frac{1}{j-1}. \tag{12}$$

We will momentarily set aside the fact that Δr_j is our decision variable and consider δ, w, and Δr_j as arbitrary variables whose sole purpose is to satisfy the equality in (12). The left side of this equation increases with δ and w, while it decreases with Δr_j. To maintain equality, a change in one of these variables must result in a change in at least one of the other two variables. There are several possible combinations of such variations, but we will emphasize three particularly important ones:

a) An increase (decrease) of δ can be compensated by a decrease (increase) of w while keeping Δr_j constant
b) An increase (decrease) of δ can be compensated by an increase (decrease) of Δr_j while keeping w constant
c) An increase (decrease) of w can be compensated by an increase (decrease) of Δr_j while keeping δ constant

These observations are crucial for understanding in which situations the marginal price for the jth unit stays constant, increases, or decreases.

Now, let us consider a specific situation: In state (t, c) with (marginal) opportunity costs $\Delta_1 V_{t-1}^{\omega}(c + 1 - j)$, we encounter a particular customer $w_{t,c}$, and calculate the corresponding optimal marginal price $\Delta r_{t,j}(c|w_{t,c})$ (based on equation (12)). The question of whether we increase (decrease) the marginal price for the jth unit in a follow-up state $(t - 1, c - i)$, $i < j$, where (marginal) opportunity costs are $\Delta_1 V_{t-2}^{\omega}(c - i + 1 - j)$, ultimately depends on the future stochastic customer $w_{t-1,c-i}$ we will face.

Hence, we search for the specific customer type w where we maintain optimality of the same marginal price $\Delta r_{t,j}(c|w_{t,c})$ in the follow-up state $(t - 1, c - i)$. With equation (12), w must fulfill:

$$\frac{\Delta_1 V_{t-2}^{\omega}(c - i + 1 - j)}{(j - 1) \cdot \Delta r_{t,j}(c|w_{t,c})} + \frac{1}{h_\lambda\left(\left(\frac{\Delta r_{t,j}(c|w_{t,c})}{w}\right)^{\frac{1}{j-1}}\right) \cdot \left(\frac{\Delta r_{t,j}(c|w_{t,c})}{w}\right)^{\frac{1}{j-1}}} = \frac{1}{j - 1}.$$

Certainly, the possibility of solving this equation in closed-form with respect to w is heavily contingent on the failure rate h_λ, and consequently, on the distribution function of λ. Distributions featuring a simple failure rate, such as the uniform distribution, allow us to formulate a closed-form expression for such a w. However, achieving this for every distribution is not possible.

Nevertheless, we can still glean some insights into the characteristics of a scenario where $\Delta r_{t,j}(c|w_{t,c})$ maintains optimality in a follow-up state: As mentioned earlier (observation a) from above), we observed that an increase (decrease) of opportunity costs can be offset by an appropriate decrease (increase) of w without altering Δr_j. The formal description of this observation is provided in the following lemma.

Lemma 3 *For $i + j \leq c$, it holds*

$$\Delta_1 V_{t-2}^{\omega}(c - i + 1 - j) \geq \Delta_1 V_{t-1}^{\omega}(c + 1 - j)$$
$$\Leftrightarrow \left(\Delta r_{t-1,j}(c - i|w_{t-1,c-i}) = \Delta r_{t,j}(c|w_{t,c}) \Rightarrow w_{t-1,c-i} \leq w_{t,c}\right)$$

Proof: This lemma is a formal description of the previous discussion and its results. □

In this section, our discussion has revolved around a scenario where the current customer $w_{t,c}$ is observed. Now, let us strive for a more comprehensive understanding of the dynamics of a marginal price that is not contingent on the observation of customer $w_{t,c}$.

These dynamics are inherently stochastic since marginal prices in both (t,c) and $(t-1,c-i)$ hinge on the arrival of two independent customers, $w_{t,c}$ and $w_{t-1,c-i}$, with their values being (ex ante) unknown. Nonetheless, we can quantify the probability of marginal prices increasing or decreasing. For the sake of simplicity, let us assume $\Delta_1 V_{t-2}^{\omega}(c-i+1-j) \geq \Delta_1 V_{t-1}^{\omega}(c+1-j)$. Drawing from observation b), we know that $\Delta r_{t-1,j}(c-i|w_{t,c}) \geq \Delta r_{t,j}(c|w_{t,c})$ for any $w_{t,c}$. This implies that the marginal unit price in a follow-up state increases compared to the current state, given the same costumer type $w_{t,c}$ in both states. Observation c) further establishes that $\Delta r_{t-1,j}(c-i|w) \geq \Delta r_{t-1,j}(c-i|w_{t,c})$ for every $w \geq w_{t,c}$ (with constant $\delta = \Delta_1 V_{t-2}^{\omega}(c-i+1-j)$). This indicates that if the customer type in the follow-up state increases compared to the current one, the marginal unit price increases even further. In summary, it follows $w_{t-1,c-i} \geq w_{t,c} \Rightarrow \Delta r_{t-1,j}(c-i|w_{t-1,c-i}) \geq \Delta r_{t,j}(c|w_{t,c})$, and thus,

$$\begin{aligned}\mathbb{P}\left(\Delta r_{t-1,j}(c-i|w_{t-1,c-i}) \geq \Delta r_{t,j}(c|w_{t,c})\right) &\geq \mathbb{P}(w_{t-1,c-i} \geq w_{t,c}) \\ &= \int_0^1 \left(\int_{w_{t,c}}^1 f_\omega(w_{t-1,c-i})\, dw_{t-1,c-i} \right) f_\omega(w_{t,c})\, dw_{t,c} \\ &= \int_0^1 \left(1 - F_\omega(w_{t,c})\right) f_\omega(w_{t,c})\, dw_{t,c} = 1 - \int_0^1 F_\omega(w_{t,c})\, f_\omega(w_{t,c})\, dw_{t,c} \\ &= 1 - \left[\frac{F_\omega(w_{t,c})^2}{2}\right]_0^1 = \frac{1}{2}.\end{aligned}$$

This result implies that the probability of the marginal price in the follow-up state being greater than or equal to the marginal price in the current state is at least $\frac{1}{2}$. Analogously, we can conclude that $\mathbb{P}\left(\Delta r_{t-1,j}(c-i|w_{t-1,c-1}) \leq \Delta r_{t,j}(c|w_{t,c})\right) \leq \frac{1}{2}$ if $\Delta_1 V_{t-2}^{\omega}(c-i+1-j) \leq \Delta_1 V_{t-1}^{\omega}(c+1-j)$.

We are now poised to consolidate all the dynamics related to the optimal pricing policy. Theorem 1 a) and b) naturally follow from the already outlined dynamics of opportunity costs. They show how optimal marginal prices, quoted to the same customer type w, change with t and c, respectively. Theorem 1 c) states that customers with a higher base willingness-to-pay encounter higher marginal prices. Lastly, Theorem 1 d) takes the stochasticity of ω into account. It states that it is more likely for marginal prices to decrease from one state to a follow-up state if the opportunity costs are higher in the follow-up state.

Theorem 1 *For every c, t, j and w, it holds:*

a) $\Delta r_{t,j}(c|w)$ *is increasing in* t, $N_{t,c}(w)$ *is decreasing in* t

b) $\Delta r_{t,j}(c|w)$ *is decreasing in* c, $N_{t,c}(w)$ *is increasing in* c

c) $\Delta r_{t,j}(c|w)$ *is increasing in* w, $N_{t,c}(w)$ *is increasing in* w

d) $\Delta_1 V_{t-2}^{\omega}(c-i+1-j) \geq \Delta_1 V_{t-1}^{\omega}(c+1-j) \Leftrightarrow \mathbb{P}\left(\Delta r_{t-1,j}(c-i|w_{t-1}) \geq \Delta r_{t,j}(c|w_t)\right) \geq \frac{1}{2}$

Proof: In observation b) regarding (12), we established marginal prices are increasing in marginal opportunity costs. Thus, a) and b) immediately follow by Propositions 3 and 4.

c) This is merely a repetition of observation c) regarding (12).

d) Proof can be found above Theorem 1 □

4.2 Observable consumption indicator

We now assume the firm is able to observe the value of next customer's consumption indicator, i.e. realization l of random variable λ is known at the moment the firm decides upon prices. Through this partially revealed information about customers' preferences, the corresponding choice behavior can be more accurately assessed. Nevertheless, there is still uncertainty present as the base willingness-top-pay ω is still stochastic.

In this section, we follow the same structure as in the previous section: we will discuss the implications of the observable consumption indicator on our customer choice model, reduce the action space to exclude irrelevant prices, solve the resulting optimization model, and show several structural properties regarding optimization model und optimal policy.

In the following, we will often face a similar structure and use similar arguments as with observable base willingness-to-pay. Whenever possible, we try to keep our explanations brief and focus more on the differences. In particular, the characteristics related to the value function, opportunity costs, and dynamics of optimal marginal prices remain consistent when considering an observable consumption indicator. Therefore, Propositions 6 and 7 convey analogous insights to Propositions 3 and 4, respectively. Similarly, Theorem 2 aligns with the conclusions drawn in Theorem 1. Consequently, we will omit detailed explanations and discussion, and generally refer to Section 4.1. However, it is important to note that Propositions 6 and 7, as well as Theorem 2, necessitate specific, new proofs due to alterations in the mathematical formulation.

4.2.1. Customer choice and model formulation

We have seen in the previous section that selling at least one unit is a deterministic occurrence with observable base willingness-to-pay. This does not transfer to a setting where the consumption indicator is observable. Every decision a customer might make is now stochastic. To ease notation, we solely focus on customers with $l > 0$.

Aiming at utility maximization, necessary conditions for purchasing j units are:

a) $\omega \geq \frac{r_j}{\sum_{k=0}^{j-1} l^k}$,

b) $\omega \geq \frac{r_j - r_i}{\sum_{k=i}^{j-1} l^k}$ for all $i \in \{1, 2, \dots, j-1\}$, and

c) $\omega \leq \frac{r_i - r_j}{\sum_{k=j}^{i-1} l^k}$ for all $i \in \{j+1, j+2, \dots, c\}$.

Analogue considerations as in Section 4.1.1 lead to the definition of relevant prices:

Lemma 4 *Relevant prices* $\boldsymbol{r}$ *are given by*

$$\mathcal{R}_c(l) = \left\{\boldsymbol{r} \in \mathbb{R}^c : 0 \leq \frac{r_j - r_{j-1}}{l^{j-1}} \leq \frac{r_{j+1} - r_j}{l^j} \leq 1 \text{ for } 1 \leq j \leq c-1\right\}.$$

Proof: With r_j such that $\frac{r_j - r_{j-1}}{l^{j-1}} = \frac{r_{j+1} - r_j}{l^j}$, there is only one customer type that is considering purchasing j units: the one with realizations w, l such that $w = \frac{r_{j+1} - r_j}{l^j}$. As ω is continuously distributed, the probability of arrival of exactly this customer type is zero. This is sufficient for the purpose of maximizing expected revenue. The same argumentation applies for $j = c$ with $\frac{r_c - r_{c-1}}{l^{c-1}} = 1$. □

Similarly to our exploration following Lemma 1, we note that $\frac{r_j - r_{j-1}}{l^{j-1}}$ represents the minimum value of realization $w \in [0,1]$ where a customer would have nonnegative marginal utility for purchasing the jth unit. This threshold is important and we define

$$\underline{w}_j(r_j - r_{j-1}) = \frac{r_j - r_{j-1}}{l^{j-1}} \qquad \text{for } j = 1, \dots, c \tag{13}$$

As established in Lemma 4, these thresholds separate the interval $[0, 1]$ in well-defined and ordered segments $[\underline{w}_j(r_j - r_{j-1}), \underline{w}_{j+1}(r_{j+1} - r_j)]$, $0 \leq j \leq c$ (setting $\underline{w}_0(r_0 - r_{-1}) = 0$ and $\underline{w}_{c+1}(r_{c+1} - r_c) = 1$). By definition, and employing the same arguments that led to (9), any customer with $w \in [\underline{w}_j(r_j - r_{j-1}), \underline{w}_{j+1}(r_{j+1} - r_j)]$ achieves maximum utility when purchasing j units.

To illustrate this point, consider the following scenario: Imagine a customer with a specific observable consumption indicator (e.g., $l = 0.8$) and an unobservable base willingness-to-pay ω with realizations $w \in [0, 1]$. The firm quotes an arbitrary price vector $\boldsymbol{r}$ with $\boldsymbol{r} \in \mathcal{R}_c(w)$ (for instance, the same price vector depicted in Figure 2). In Figure 4, we portray the marginal utility as a function of every possible realization $w \in [0, 1]$, i.e., $l \mapsto w \cdot l^{j-1} - (r_j - r_{j-1})$. We only display the marginal utilities for $j \leq 3$.

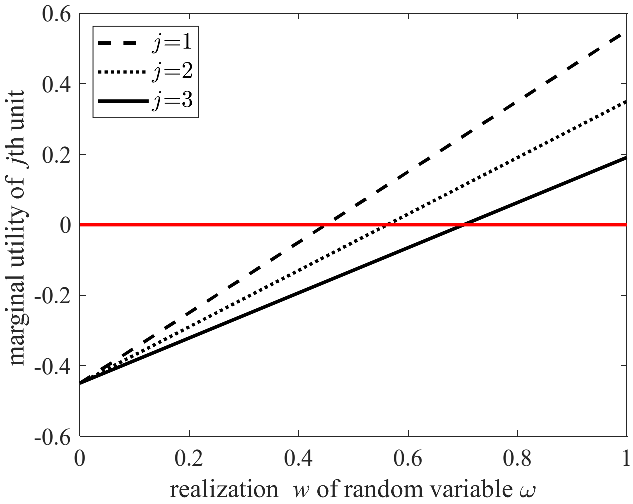

Figure 4: Exemplary marginal utility for $j \leq 3$ depending on realization w

The marginal utility is a linear function in w with gradient l^{j-1}. Moreover, the marginal utility at $w = 0$ is $-(r_j - r_{j-1})$. As we employed a linear pricing scheme in this example, every of the portrayed lines start at $-(r_j - r_{j-1}) = -0.45$. We can clearly observe that the interval $[0, 1]$ (at the red line) is separated in four intervals, $[0, \underline{w}_1(r_1 - r_0)]$, $[\underline{w}_1(r_1 - r_0), \underline{w}_2(r_2 - r_1)]$, $[\underline{w}_2(r_2 - r_1), \underline{w}_3(r_3 - r_2)]$, and $[\underline{w}_3(r_3 - r_2), 1]$.

Just like the derivation of (9), restricting the action space on $\mathcal{R}_c(l)$ has the advantage that the probability of selling j units simplifies to:

$$p_j(\boldsymbol{r}|l) = \begin{cases} F_\omega\left(\underline{w}_{j+1}(r_{j+1} - r_j)\right) - F_\omega\left(\underline{w}_j(r_j - r_{j-1})\right) & \text{for } 1 \leq j \leq c-1 \\ 1 - F_\omega\left(\underline{w}_c(r_c - r_{c-1})\right) & \text{for } j = c \end{cases}$$

The optimization problem with observable consumption indicator is given by

$$V_t^\lambda(c) = \int_0^1 \max_{\boldsymbol{r} \in \mathcal{R}_c(l)} \left\{ \sum_{j=1}^{c} p_j(\boldsymbol{r}|l) \cdot \left(r_j - \Delta_j V_{t-1}^\lambda(c)\right) \right\} \cdot f_\lambda(l)\, dl + V_{t-1}^\lambda(c) \tag{14}$$

with boundary conditions $V_0^\lambda(c) = 0$ for $c \geq 0$ and $V_t^\lambda(0) = 0$ for $t \geq 0$. $V_t^\lambda(c)$ is the optimal expected revenue-to-go from period t onwards (before observing the customer in t). In contrast to the general setting, the firm has access to realization l of customers' consumption indicator λ before quoting prices. For every possible l, we denote the corresponding optimal batch prices selected in state (t, c) by $\boldsymbol{r}_t(c|l) \in \mathcal{R}_c(l)$.

4.2.2. Solution and structural properties

The maximum number of units we can economically sell depends on the state (t, c) and the realized consumption indicator l.

Corollary 2 *If* $V_{t-1}^{\lambda}(\cdot)$ *is increasing and concave,* $N_{t,c}(l) = \max_{j=1,\dots,c} \{j: \Delta_1 V_{t-1}^{\lambda}(c-j+1) < l^{j-1}\}$ *denotes the highest number of units that can be economically sold. It holds that* $\{j: \Delta_1 V_{t-1}^{\lambda}(c-j+1) < l^{j-1}\} = \{1, 2, \dots, N_{t,c}(l)\}$.

Proof: $\Delta_1 V_{t-1}^{\lambda}(c-j+1)$ is increasing in j, while l^{j-1} is decreasing. □

In the following, we ignore selling more than $N_{t,c}(l)$ units. Technically, we choose prices r_j for $j > N_{t,c}(l)$ sufficiently large such that no sell occurs almost surely. This holds, e.g., for $r_j = l^{j-1} + r_{j-1}$.

By definition, we have $\boldsymbol{\Delta_1 V_{t-1}^{\lambda}\big(c - N_{t,c}(l) + 1\big) < l^{N_{t,c}(l)-1}}$ and $\boldsymbol{\Delta_1 V_{t-1}^{\lambda}\big(c - \big(N_{t,c}(l) + 1\big) + 1\big) \geq l^{N_{t,c}(l)}}$. Consequently, it follows that $\boldsymbol{\Delta_1 V_{t-1}^{\lambda}\Big((c-1) - \big(N_{t,c}(l) - 1\big) + 1\Big) < l^{N_{t,c}(l)-1}}$ and $\boldsymbol{\Delta_1 V_{t-1}^{\lambda}\Big((c-1) - N_{t,c}(w) + 1\Big) \geq l^{N_{t,c}(l)}}$, respectively. Utilizing $\boldsymbol{\Delta_1 V_{t-1}^{\lambda}\Big((c-1) - \big(N_{t,c}(l) - 1\big) + 1\Big) < l^{N_{t,c}(l)-1} < l^{(N_{t,c}(l)-1)-1}}$, we can conclude that $\boldsymbol{N_{t,c}(l) - 1 \in \{j: \Delta_1 V_{t-1}^{\lambda}(c - j + 1) < l^{j-1}\}}$, thus establishing $\boldsymbol{N_{t,c-1}(l) \geq N_{t,c}(l) - 1}$. Similarly, with $\boldsymbol{\Delta_1 V_{t-1}^{\lambda}\Big((c-1) - \big(N_{t,c}(w) + 1\big) + 1\Big) \geq \Delta_1 V_{t-1}^{\lambda}\Big((c-1) - N_{t,c}(w) + 1\Big) \geq l^{(N_{t,c}(l)+1)-1}}$, we deduce that $\boldsymbol{N_{t,c}(l) + 1 \notin \{j: \Delta_1 V_{t-1}^{\lambda}(c - j + 1) < l^{j-1}\}}$, indicating that $\boldsymbol{N_{t,c-1}(l) < N_{t,c}(l) + 1}$. Consequently, $\boldsymbol{N_{t,c-1}(l)}$ either equals $\boldsymbol{N_{t,c}(l) - 1}$ or $\boldsymbol{N_{t,c}(l)}$. This observation leads to the following remark.

Remark 3 *It holds that* $N_{t,c-1}(l) \leq N_{t,c}(l) \leq N_{t,c-1}(l) + 1$.

Unlike the previous section where the base willingness-to-pay was observable, there are now two cases to consider for the maximal number of units sold in adjacent states. This introduces additional complexity in our upcoming proofs.

Proposition 5 *If* $V_{t-1}^{\lambda}(c)$ *is increasing and concave in* c*, it holds: In every state* (t, c) *and for every* $l \in [0,1]$ *the optimal marginal price* $\Delta r_{t,j}(c|l)$ *for the* j*th unit,* $j = 1, \dots, N_{t,c}(l)$*, is given by* $\Delta r_{t,j}(c|l) = l^{j-1} \cdot \underline{w}_j$ *with* $\underline{w}_j$ *such that* $l^{j-1} \cdot \left(\underline{w}_j - \frac{1}{h_\omega(\underline{w}_j)}\right) = \Delta_1 V_{t-1}^{\lambda}(c + 1 - j)$.

Proof: See Supplement S.3.

Remark 4 *For* $w \sim U[0,1]$*, the optimality condition leads to a closed-form solution:*

$$\Delta r_{t,j}(c|l) = \frac{1}{2} \cdot \left(l^{j-1} + \Delta_1 V_{t-1}^{\lambda}(c + 1 - j)\right).$$

Remark 5 *The pricing structure divides customers with the same consumption indicator into groups based on their base willingness-to-pay. The higher a customer's willingness-to-pay, the more units are being sold. Specifically, a base willingness-to-pay of* $r_{t,1}$ *separates customers who buy nothing at all and customers who purchase at least one unit.*

Remark 6 *In the supplement (namely S.3), we show that Lemma 2 carries over to Section 4.2. Hence, it still holds that the expected margin for selling the* j*th unit is greater than the expected margin for selling the* $j+1$*th unit.*

So far, we found the optimal solution in period t under the condition that the value function in period $t-1$ is increasing and concave in c. We will now proof that this condition indeed holds for the whole selling horizon.

Proposition 6 *For every* t, $V_t^\lambda(\cdot)$ *is increasing and concave.*

Proof: See Supplement S.5.

In addition to Proposition 6, further structural properties of value function $V_t^\lambda(\cdot)$ and resulting opportunity costs $\Delta_1 V_t^\lambda(c)$ are given by the following proposition:

Proposition 7 *For every* c*, it holds:*

a) $\Delta_1 V_t^\lambda(c)$ *is increasing in* t

b) $V_t^\lambda(c)$ *is increasing and concave in* t

Proof: See Supplement S.6.

We have seen in the previous section that dynamics of opportunity costs are an important driver to pricing dynamics. Similarly, based on the optimality condition $\Delta r_j - \frac{1}{h_\omega\left(\frac{\Delta r_j}{l j-1}\right)} = \delta$, it again holds that optimal marginal prices Δr_j are increasing in customer type l and in opportunity costs δ.

Theorem 2 *For every* c, t, j *and* l*, it holds:*

a) $\Delta r_{t,j}(c|l)$ *is increasing in* t*,* $N_{t,c}(l)$ *is decreasing in* t

b) $\Delta r_{t,j}(c|l)$ *is decreasing in* c*,* $N_{t,c}(l)$ *is increasing in* c

c) $\Delta r_{t,j}(c|l)$ *is increasing in* l*,* $N_{t,c}(l)$ *is increasing in* l

d) $\Delta r_{t,1}(c|l)$ *is independent of* l

e) $\Delta_1 V_{t-2}^\lambda(c-i+1-j) \geq \Delta_1 V_{t-1}^\lambda(c+1-j) \Rightarrow \mathbb{P}\left(\Delta r_{t-1,j}(c-i|l_{t-1}) \geq \Delta r_{t,j}(c|l_t)\right) \geq \frac{1}{2}$

Proof: a) – d) are immediate results of Propositions 5, 6, and 7.

e) holds with $\Delta_1 V_{t-2}^\lambda(c-i+1-j) \geq \Delta_1 V_{t-1}^\lambda(c+1-j)$

$$\Rightarrow \mathbb{P}\left(\Delta r_{t-1,j}(c-i|l_{t-1,c-1}) \geq \Delta r_{t,j}(c|l_{t,c})\right) \geq \mathbb{P}(l_{t-1,c-1} \geq l_{t,c})$$

$$= \int_0^1 \left(\int_{l_{t,c}}^1 f_\lambda(l_{t-1,c-1})\, dl_{t-1,c-1} \right) f_\lambda(l_{t,c})\, dl_{t,c} = \int_0^1 \left(1 - F_\lambda(l_{t,c})\right) f_\lambda(l_{t,c})\, dl_{t,c}$$

$$= 1 - \int_0^1 F_\lambda(l_{t,c})\, f_\lambda(l_{t,c})\, dl_{t,c} = 1 - \left[\frac{F_\lambda(l_{t,c})^2}{2}\right]_0^1 = \frac{1}{2}$$

□

4.2.3. Special case: Uniform Distribution

In Remark 5, we provided the closed-form expression of optimal marginal prices. This allows us to compute selling probabilities and the expected revenue, $V_t^\lambda(c|l)$, for every possible realization l of random consumption indicator λ. Subsequently, these l-dependent expected revenues can be employed to calculate the overall expected revenue $V_t^\lambda(c)$ of state (t,c):

$$
\begin{aligned}
V_t^\lambda(c) = \frac{1}{4}\cdot\Bigg(&\left(1-\Delta_1 V_{t-1}^\lambda(c)\right)^2 + \frac{1}{2} - 2\Delta_1 V_{t-1}^\lambda(c-1) \\
&+ \left(\frac{3}{2} - \ln\left(\Delta_1 V_{t-1}^\lambda(c-1)\right)\cdot\left(\Delta_1 V_{t-1}^\lambda(c-1)\right)^2\right) \\
&+ \sum_{j=3}^{c}\Bigg(\frac{1}{j} - 2\Delta_1 V_{t-1}^\lambda(c+1-j) - \frac{\left(\Delta_1 V_{t-1}^\lambda(c+1-j)\right)^2}{j-2} \\
&+ \frac{2(j-1)^2}{j(j-2)}\left(\Delta_1 V_{t-1}^\lambda(c+1-j)\right)^{\frac{1}{j-1}}\Bigg)\Bigg) + V_{t-1}^\lambda(c)
\end{aligned}
$$

Moreover, we want to point out the special structure of $r_{t,j}(c|l)$: Consisting of $\sum_{k=0}^{j-1} l^k$ and $\Delta_j V_{t-1}^\lambda(c)$, $r_{t,j}(c|l)$ is increasing in j. While the first component is apparently concave in j ($l \in [0,1]$), the second component is convex in j (as $\Delta_j V_{t-1}^\lambda(c) = \sum_{k=1}^{j} \Delta_1 V_{t-1}^\lambda(c+1-k)$ and $\Delta_1 V_{t-1}^\lambda(c-k)$ is increasing in k (cf. Proposition 6)).

4.3 Observable base willingness-to-pay and consumption indicator

In this section, we assume that a firm can observe next customer's base willingness-to-pay and consumption indicator, i.e. realizations w and l of random variables ω and λ, respectively, are known when the firm decides upon prices. Thereby, we eliminate every stochasticity of customers' behavior and the whole optimization problem becomes deterministic:

$$
p_j(\boldsymbol{r}|w,l) = 1_{\left\{\max\limits_{i=1,\dots,c}\left\{w\cdot\sum_{k=0}^{i-1} l^k - r_i, 0\right\} = w\cdot\sum_{k=0}^{j-1} l^k - r_j\right\}}, \quad 1 \le j \le c
$$

$$
p_0(\boldsymbol{r}|w,l) = 1_{\left\{\max\limits_{i=1,\dots,c}\left\{w\cdot\sum_{k=0}^{i-1} l^k - r_i\right\} < 0\right\}},
$$

for $\boldsymbol{r} \in \mathcal{R}_c(w,l) = \left\{\boldsymbol{r} \in \mathbb{R}^c : p_0(\boldsymbol{r}|w,l) + \sum_{j=1}^{c} p_j(\boldsymbol{r}|w,l) = 1\right\}$. Restricting the action space to $\mathcal{R}_c(w,l)$ is a technical decision to make the $p_j(\boldsymbol{r}|w,l)$ work the way it is intended. Otherwise, we would allow for selling a single customer every batch size at once by setting $r_j = w\cdot\sum_{i=0}^{j-1} l^i$ for every j. Alternatively, we could use a more elaborate definition of $p_j(\boldsymbol{r}|w,l)$ together with a set of assumptions regarding tiebreakers when a customer faces equally good options. As both ways have the same outcome, we preferred to have a simple definition of $p_j(\boldsymbol{r}|w,l)$.

The optimization problem is given by:

$$V_t^{\omega,\lambda}(c) = \int_0^1 \int_0^1 \max_{\boldsymbol{r}\in\mathcal{R}_c(w,l)} \left\{\sum_{j=1}^{c} p_j(\boldsymbol{r}|w,l) \cdot \left(r_j - \Delta_j V_{t-1}^{\omega,\lambda}(c)\right)\right\} \cdot f_\lambda(l) f_\omega(w)\, dl\, dw + V_{t-1}^{\omega,\lambda}(c)\,, \tag{15}$$

with boundary conditions $V_0^{\omega,\lambda}(c) = 0$ for $c \geq 0$ and $V_t^{\omega,\lambda}(0) = 0$ for $t \geq 0$. Note that we still calculate expected revenue even though maximizing is now deterministic.

Without eliminating demand, the highest possible batch price r_j for j units is $r_j = w \cdot \sum_{i=0}^{j-1} l^i$. Thus, we are looking for the batch size with the highest possible additional revenue, i.e. $j^* = \arg\max_{1\leq k\leq c}\{w \cdot \sum_{i=0}^{k-1} l^i - \Delta_k V_{t-1}^{\omega,\lambda}(c)\}$. If $w \cdot \sum_{i=0}^{j^*-1} l^i - \Delta_{j^*} V_{t-1}^{\omega,\lambda}(c) < 0$, we are not able to economically sell something to the current customer. In this case, we prefer not selling anything and pick $r_j > w \cdot \sum_{i=0}^{j-1} l^i$ for every j. If $w \cdot \sum_{i=0}^{j^*-1} l^i - \Delta_{j^*} V_{t-1}^{\omega,\lambda}(c) \geq 0$, we can earn additional revenue. By setting $r_{j^*} = w \cdot \sum_{i=0}^{j^*-1} l^i$ and $r_j > w \cdot \sum_{i=0}^{j-1} l^i$, $j \neq j^*$, we ensure $\boldsymbol{r} \in \mathcal{R}_c(w,l)$ and have the optimal solution for given w, l.

Lemma 5 *For every w, l, the best batch size greater than zero is given by*

$$j = arg \max_{1\leq k\leq c}\left\{w \cdot \sum_{i=0}^{k-1} l^i - \Delta_k V_{t-1}^{\omega,\lambda}(c)\right\}.$$

The optimal solution to the maximization in (15) is given by:

- $r_{t,j}(c|w,l) = w\sum_{i=0}^{j-1} l^i$ *and* $r_{t,k}(c|w,l) > w\sum_{i=0}^{k-1} l^i$, $k \neq j$, *if* $w\sum_{i=0}^{j-1} l^i - \Delta_j V_{t-1}^{\omega,\lambda}(c) \geq 0$
- $r_{t,k}(c|w,l) > w\sum_{i=0}^{k-1} l^i$ *for every* k, *if* $w\sum_{i=0}^{j-1} l^i - \Delta_j V_{t-1}^{\omega,\lambda}(c) < 0$

Proof: above Lemma 5.

Even though solving the maximization problem is trivial, calculating $V_t^{\omega,\lambda}(c)$ is not. There are many cases to consider, and thus, it is not easy to find for every unit size j the subset of $(w,l) \in [0,1]^2$ where $j = \arg\max_{1\leq k\leq c}\{w \cdot \sum_{i=0}^{k-1} l^i - \Delta_k V_{t-1}^{\omega,\lambda}(c)\}$ as well as $w \cdot \sum_{i=0}^{j-1} l^i - \Delta_j V_{t-1}^{\omega,\lambda}(c) \geq 0$.

Again, it is useful to look at marginal prices and opportunity costs: For $j = \arg\max_{1\leq k\leq c}\{w \cdot \sum_{i=0}^{k-1} l^i - \Delta_k V_{t-1}^{\omega,\lambda}(c)\}$ it holds that $w \cdot l^{j-1} \geq \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)$ and $w \cdot l^j < \Delta_1 V_{t-1}^{\omega,\lambda}(c-j)$. For the time being, this is a necessary but no sufficient condition on $(w,l) \in [0,1]^2$. It only ensures that selling j units is better than selling $j-1$ and $j+1$ units. Neither does it automatically make j the best batch size nor does it ensure the firm is earning additional revenue, i.e. $w \cdot \sum_{i=0}^{j-1} l^i - \Delta_j V_{t-1}^{\omega,\lambda}(c) \geq 0$. Looking at the aforementioned necessary condition, we observe $w \cdot l^{i-1} \geq w \cdot l^{j-1} \geq \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)$, $i \leq j$, and

$w \cdot l^i < w \cdot l^j < \Delta_1 V_{t-1}^{\omega,\lambda}(c-j)$, $i > j$. Assuming a suitable structure of $\Delta_1 V_{t-1}^{\omega,\lambda}(\cdot)$, we can derive the following lemma:

Lemma 6 *If $V_{t-1}^{\omega,\lambda}(\cdot)$ is increasing and concave, it holds: j units is the optimal batch size to sell to every customer with $(w,l) \in [0,1]^2$ such that $w \cdot l^{j-1} \geq \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)$, and $w \cdot l^j < \Delta_1 V_{t-1}^{\omega,\lambda}(c-j)$. We define this number by $N_{t,c}(w,l) = \max_{j=1,\ldots,c} \{j: \Delta_1 V_{t-1}^{\omega,\lambda}(c-j+1) \leq w \cdot l^{j-1}\}$.*

Proof: $V_{t-1}^{\omega,\lambda}(\cdot)$ is increasing and concave, thus $\Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)$ is increasing in j. With $(w,l) \in [0,1]^2$ such that $w \cdot l^{j-1} \geq \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)$, it holds:

$$w \cdot l^{i-1} \geq w \cdot l^{j-1} \geq \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j) \geq \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-i), \qquad i \leq j,$$

and

$$w \cdot l^i \leq w \cdot l^j < \Delta_1 V_{t-1}^{\omega,\lambda}(c-j) \leq \Delta_1 V_{t-1}^{\omega,\lambda}(c-i), \qquad i > j.$$

Finally, we can conclude $\max_{1 \leq k \leq c} \{w \cdot \sum_{i=0}^{k-1} l^i - \Delta_k V_{t-1}^{\omega,\lambda}(c)\} = w \cdot \sum_{i=0}^{j-1} l^i - \Delta_j V_{t-1}^{\omega,\lambda}(c) \geq 0$ making j the optimal batch size to sell. □

Remark 7 *In Sections 4.1 and 4.2, $N_{t,c}$ served as an upper bound on the number of units a firm could sell economically, a consequence of the uncertainty arising from the unobservable part of customers' information. During these instances, the firm lacked precise knowledge regarding the actual number of units it might sell to a current customer, but it recognized that overall expected revenues could be optimized by selling up to $N_{t,c}$ units. However, in this section, stochasticity is entirely eliminated, and the firm is fully aware of the quantity of units it sells for a given price. Therefore, $N_{t,c}$ precisely denotes the number of units a firm sells to a customer to maximize overall expected revenues.*

Proof of Lemma 6 also showed that a firm sells in optimality at least j units to a customer with $(w,l) \in [0,1]^2$ such that $w \cdot l^{j-1} \geq \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)$. This implies that every such customer is purchasing the jth unit. We can make use of this observation to show concavity of $V_t^{\omega,\lambda}(\cdot)$ and concentrate on $V_t^{\omega,\lambda}(\cdot\,|w,l) = \sum_{j=1}^{c} 1_{\{w \cdot l^{j-1} \geq \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)\}} \cdot \left(w \cdot l^{j-1} - \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)\right) + V_{t-1}^{\omega,\lambda}(\cdot)$ for every realization w, l. In the proof of concavity, we need the following property regarding $N_{t,c}(w,l)$, the number of units a certain customer is purchasing.

Lemma 7 *If $V_{t-1}^{\omega,\lambda}(\cdot)$ is increasing and concave, for every $(w,l) \in [0,1]^2$, it holds that*

$$N_{t,c+1}(w,l) - 1 \leq N_{t,c}(w,l) \leq N_{t,c+1}(w,l).$$

Proof: $V_{t-1}^{\omega,\lambda}(\cdot)$ is increasing and concave, thus $\Delta_1 V_{t-1}^{\omega,\lambda}\left(c+1-N_{t,c}(w,l)\right) \geq \Delta_1 V_{t-1}^{\omega,\lambda}\left(c+2-N_{t,c}(w,l)\right)$. Together with $w \cdot l^{N_{t,c}(w,l)-1} \geq \Delta_1 V_{t-1}^{\omega,\lambda}\left(c+1-N_{t,c}(w,l)\right)$, it holds that $N_{t,c}(w,l) \leq N_{t,c+1}(w,l)$. Based on $w \cdot l^{N_{t,c}(w,l)+1} \leq w \cdot l^{N_{t,c}(w,l)} < \Delta_1 V_{t-1}^{\omega,\lambda}\left(c-N_{t,c}(w,l)\right)$, it also holds that

$N_{t,c}(w,l) + 2 > N_{t,c+1}(w,l)$. As $N_{t,c}(w,l)$ and $N_{t,c+1}(w,l)$ are integer, we can use $N_{t,c}(w,l) + 2 \geq N_{t,c+1}(w,l) + 1$ instead. □

We now have everything to state and show the following proposition.

Proposition 8 $V_t^{\omega,\lambda}(\cdot)$ *is increasing and concave.*

Proof: See Supplement S.7.

Other dynamics of opportunity costs and value function are given in the following proposition.

Proposition 9 *For every* c*, it holds:*

a) $\Delta_1 V_t^{\omega,\lambda}(c)$ *is increasing in* t

b) $V_t^{\omega,\lambda}(c)$ *is increasing and concave in* t

Proof: See Supplement S.8.

Proposition 9 has an immediate implication on dynamics of optimal marginal prices: As $\Delta_1 V_t^{\omega,\lambda}(c)$ is increasing in t, it is less likely that a customer arrives with $w \cdot l^{j-1} \geq \Delta_1 V_t^{\omega,\lambda}(c+1-j)$ for higher t. Thereby, the probability of selling the jth unit $\mathbb{P}\left(w \cdot l^{j-1} \geq \Delta_1 V_t^{\omega,\lambda}(c+1-j)\right)$ decreases. Moreover, as selling j units is increasingly restricted to customers with high w and l in the optimal solution, the average price $r_{t,j}(c)$ that can be earned by selling j units increases.

We conclude this section with a summary of all dynamics regarding optimal marginal prices we found.

Theorem 3 *For every* c, t, w*, and* l*, it holds:*

a) $r_{t,j}(c|w,l)$ *is constant in* t *as long as* $j = N_{t,c}(w,l)$*,* $N_{t,c}(w,l)$ *is decreasing in* t

b) $r_{t,j}(c)$ *is increasing in* t *for every* j

c) $r_{t,j}(c|w,l)$ *is constant in* c *as long as* $j = N_{t,c}(w,l)$*,* $N_{t,c}(w,l)$ *is increasing in* c

d) $r_{t,j}(c)$ *is decreasing in* c *for every* j

e) $r_{t,j}(c|w,l)$ *is increasing in* w *and* l *for every* j*,* $N_{t,c}(w,l)$ *is increasing in* w *and* l

Proof: a) – e) follow by Lemma 5, Lemma 6, Proposition 8, and Proposition 9. □

In light of Theorem 3, it is evident that a firm maintains the same price for two customers with identical w and l in adjacent states as long as the optimal batch size $N_{t,c}(w,l)$ remains unchanged (refer to a) and c)). However, the optimal batch size tends to decrease over time and increase with capacity. Essentially, the scarcer the product, the smaller the optimal batch size. Additionally, the firm quotes higher prices to customers with higher w or l and tends to increase the offered batch size (cf., e)).

Moreover, we've observed that the average price quoted by a firm for j units increases with t and decreases with c (cf., b) and d)). Understanding the dynamics of average prices is advantageous as they are not contingent on a specific customer represented by w and l. In any selling process, the realization of a customer stream with specific w_t and l_t can lead to counterintuitive price changes (such as raising prices even if the firm did not sell in the previous period). However, on average, the optimal policy adheres to the conventional intuitive structure where prices increase if the product becomes scarcer (due to an increase of t or decrease of c).

5 Simulation Study

In this section, we compare earned revenues of (up to) four different kinds of observable information:

- Full information (***FI***): Observable base willingness-to-pay ω and consumption indicator λ (refer to Section 4.3).

  As there is perfect personalized pricing and deterministic customer behavior, this scenario reflects the highest possible revenues earned. We will often refer to this case as upper bound.
- Partial information (***PI-$\boldsymbol{\omega}$***): Observable base willingness-to-pay ω (refer to Section 4.1).

  In this scenario, we have no closed-form solution for the optimization problem, and thus, solve it numerically.
- Partial information (***PI-$\boldsymbol{\lambda}$***): Observable consumption indicator λ (refer to Section 4.2).

  In this scenario, we have a closed-form solution if $\omega \sim U[0, 1]$. Otherwise, we solve it numerically.
- No information (***NI***):

  We use heuristic D from Schur (2023) and describe it briefly in Section 5.1. This heuristic solves the optimization problem without observable information for $t = 1$ optimally, and for $t > 1$ approximately.

For our simulation study, we align our setting with Gallego et al. (2020). Accordingly, we set $T = 1, \dots, 40$, $C = 1, \dots, 120$, and consider $\omega, \lambda \sim U[0, 1]$. In each state, we employ a random sample of $10{,}000$ realizations for both ω and λ. Throughout Section 5, each presented revenue is derived from this randomized dataset and the corresponding policy generated by one of our mechanisms or heuristics.

In Section 5.1, we describe all three heuristics developed in Schur (2023) for the no information case (*NI*). Specifically, we elaborate on heuristic *D,* as it has proven to be the best-performing one. In Section 5.2, we determine the optimal solution for all four types of observable information: *FI*, *PI*-ω, *PI*-λ, and *NI*, across every state (t, c), $t \leq 40, c \leq 120$. The pair-wise differences in the resulting expected revenues represent the value of information. For instance, the discrepancy between the revenues of *FI* and *PI*-ω indicates the additional revenue that could be earned if both ω and λ were observable instead of only ω.

Moving to Section 5.3, we delve into the impact of the distribution of ω and λ. Alongside the uniform distribution, we opt for a (truncated) normal distribution with a mean of 0.5 and a standard deviation of

0.1. This introduces two distributions with the same mean but significantly different deviations. In Section 5.4, we relax our assumption that parameters can be precisely observed. Instead, we operate with predefined distinct intervals, assuming that the firm can accurately allocate (formerly observable) realizations to these intervals. Finally, in Section 5.5, we delve into an additional layer of decision-making. Specifically, we explore the scenario where the firm has the autonomy to determine its initial stock and investigate the implications of allowing the firm to decide on restocking in the middle of the planning horizon.

5.1 Heuristics for the no information case

The following heuristics $E(\lambda)$, $E(\omega)$, and *D* were developed in Schur (2023), and we refer to this work for a detailed analysis. However, we want to shortly explain how these heuristics work and why we chose to employ heuristic *D*.

Heuristics $E(\lambda)$ and $E(\omega)$ share the same underlying idea and rely on the results of our work. In our research, we demonstrated that we can find the optimal solution if we can observe the realization of λ (Section 4.2) or ω (Section 4.1). The optimal price vectors are dependent on the realization of these random variables, becoming random optimal price vectors. Consequently, we can build the expected value and obtain a price vector known as the expected optimal price in Schur (2023). Both heuristics differ in the realization they use to define these random optimal price vectors. $E(\lambda)$ employs the realization of λ, utilizing our work discussed in Section 4.2, while $E(\omega)$ builds on the realization of ω, stemming from our work discussed in Section 4.1.

Heuristic *D* decomposes batches into distinguished units (1st, 2nd, etc.) and separately optimizes prices for each ith unit, where $i = 1, \ldots, c$. This approach utilizes a simplified customer choice behavior and is similar to the one applied in Sections 4.1 and 4.2 to solve the optimization problem (see, e.g., (11)). However, in our case, we initially introduced this decomposition as an upper bound to our problem and later proved that it yields in the same values and optimal solutions as the original problem. This equivalence does not hold for a setting where neither random variable is observable. In such a scenario, this decomposition does not result in the same values and solutions and does not constitute an upper bound. However, in a simulation study, this heuristic yielded the highest revenues. It is worth noting that $E(\omega)$ produced almost the same revenues. This could be interpreted as an indication that both heuristics might be relatively close to the (unknown) optimal value. The choice to employ heuristic *D* in our current work was driven by its demonstrated effectiveness and higher revenue outcomes in comparison to the other two heuristics.

All three heuristics are further enhanced with the help of a fluid approximation. The fluid approximation finds the optimal solutions in states without opportunity costs (i.e., for $t = 1$). Additionally, it forms a policy that is asymptotically optimal (refer to, e.g., Schur, 2023, Maglaras & Meissner, 2006, and Gallego & van Ryzin, 1997) and transfers this property to heuristics it is combined with.

5.2 Value of Information

Table 2 shows expected revenues for all kinds of observable information with $C \in \{1, 20, 40, 60, 80, 100, 120\}$. Scenarios *FI* and *NI* are the upper and lower bound, respectively. In between, *PI-*ω is outperforming *PI-*λ in every state. For $C = 1$, there is just one unit of the product for sale, and thus, no multiunit demand can be served. In this state, *PI-*ω is performing like the full information scenario *FI*, and *PI-*λ like the no information scenario *NI*. The more capacity, the higher is the importance of attending customers' demand for more than one unit. This can be seen by comparing mechanisms *PI-*ω and *PI-*λ. While the absolute difference is increasing for $C \leq 100$, we can observe that the relative difference is shrinking between those two scenarios for $C \geq 60$.

Table 2: Revenues for $C \leq 120, T = 40$

$T = 40$	*FI*	*PI-*ω	*PI-*λ	*NI*
$C = 1$	0.96 €	0.96 €	0.91 €	0.91 €
$C = 20$	15.50 €	15.08 €	12.62 €	12.40 €
$C = 40$	26.70 €	24.53 €	20.18 €	19.36 €
$C = 60$	35.70 €	31.03 €	25.71 €	24.15 €
$C = 80$	43.29 €	35.82 €	30.06 €	27.71 €
$C = 100$	49.84 €	39.53 €	33.63 €	30.49 €
$C = 120$	55.61 €	42.50 €	36.62 €	32.74 €

To get a clear image regarding the relative value of information, we divide expected revenue of every scenario by upper bound *FI*. Thereby, we show the percentage of the best possible outcome every kind of information yields.

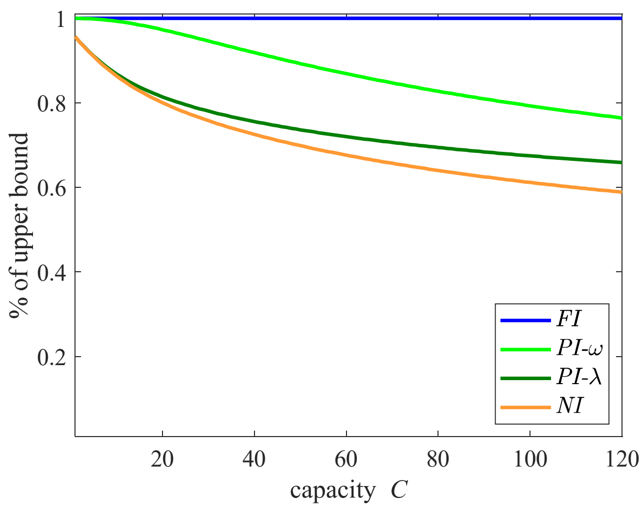

Figure 5: Performance of all mechanism relative to an upper bound for $C \leq 120, T = 40$

Figure 5 displays the same order as shown in Table 2, i.e. *FI* ≥ *PI-*ω ≥ *PI-*λ ≥ *NI*. For a lower amount of capacity ($C \leq 20$), mechanisms *FI* and *PI-*ω as well as *PI-*λ and *NI* are performing similarly with a significant gap between both groups. For a higher amount of capacity ($C \geq 40$), mechanism *FI* is significantly outperforming *PI*-ω while *PI*-λ is marginally better than *NI*. The gap between *PI*-ω and *PI*-λ is decreasing with capacity. However, it is still noticeably large.

These observations lead to the following conclusions: Observing the base willingness-to-pay is considerably more valuable than observing the consumption indicator. However, observing the consumption indicator is not useless. This information adds value in settings where the capacity is only moderately scarce or where the firm is able to also observe the base willingness-to-pay. In the latter case, the increase in revenue is especially large for higher capacity levels (ca. 30% for $C = 120$).

5.3 Different Distributions

In this section, we explore the impact of the distribution of ω and λ on expected revenues resulting from partial (*PI*-ω and *PI*-λ) and full information (*FI*) about customers' private information. We consider two different distributions: a uniform distribution (denoted as $U[0,1]$) and a (truncated) normal distribution with mean of 0.5 and standard deviation of 0.1 (denoted as $N[0.5,0.1,0,1]$). Both distributions share the same mean (0.5) but have significantly different deviations ($\sqrt{\frac{1}{12}} \approx 0.28$ vs. 0.1). We investigate every combination of ω and λ following one of the two distributions.

Table 3: Revenues for $C \leq 120, T = 40$ and different distributions

	$\omega, \lambda \in U[0,1]$			$\omega \sim U[0,1], \lambda \sim N[0.5,0.1,0,1]$		
$T = 40$	*FI*	*PI*-ω	*PI*-λ	*FI*	*PI*-ω	*PI*-λ
$C = 1$	0.96 €	0.96 €	0.91 €	0.96 €	0.96 €	0.85 €
$C = 20$	15.50 €	15.08 €	12.62 €	14.66 €	14.60 €	11.17 €
$C = 40$	26.70 €	24.53 €	20.18 €	23.09 €	22.19 €	15.75 €
$C = 60$	35.70 €	31.03 €	25.71 €	28.60 €	26.42 €	18.05 €
$C = 80$	43.29 €	35.82 €	30.06 €	32.37 €	28.86 €	19.27 €
$C = 100$	49.84 €	39.53 €	33.63 €	35.00 €	30.27 €	19.95 €
$C = 120$	55.61 €	42.50 €	36.62 €	36.85 €	31.08 €	20.33 €
	$\omega \sim N[0.5,0.1,0,1], \lambda \sim U[0,1]$			$\omega, \lambda \sim N[0.5,0.1,0,1]$		
$T = 40$	*FI*	*PI*-ω	*PI*-λ	*FI*	*PI*-ω	*PI*-λ
$C = 1$	0.69 €	0.69 €	0.65 €	0.69 €	0.69 €	0.65 €
$C = 20$	11.65 €	11.56 €	10.23 €	11.45 €	11.45 €	9.80 €
$C = 40$	21.36 €	20.79 €	17.91 €	20.13 €	20.05 €	15.80 €
$C = 60$	29.71 €	27.87 €	24.20 €	26.05 €	25.03 €	19.72 €
$C = 80$	37.05 €	33.20 €	29.52 €	30.39 €	28.03 €	22.37 €
$C = 100$	43.56 €	37.34 €	34.13 €	33.53 €	29.82 €	24.18 €
$C = 120$	49.39 €	40.66 €	38.18 €	35.80 €	30.87 €	25.41 €

Table 3 presents the results of our simulation study. One noticeable effect is that expected revenues are higher for distributions with higher deviation. This holds true for every kind of observable information (*PI*-ω, *PI*-λ, and *FI*) as well as for random variables ω and λ. However, the magnitude of this effect varies across different scenarios. A smaller deviation of ω, i.e., $\omega \sim N[0.5,0.1,0,1]$ instead of $\omega \sim U[0,1]$, has a more (less) significant impact on settings with low (high) capacity C. Conversely, for λ, we observe the opposite effect. Furthermore, the order observed in Section 5.1 is validated for every combination of distributions. Notably, for $\omega \sim N[0.5,0.1,0,1]$ and $\lambda \sim U[0,1]$, *PI*-λ is very close to *PI*-ω, and the gap between both mechanisms diminishes for higher C.

To provide a clearer overview of the influence of different distributions on different kinds of observable information, we depict the relative performances of *PI*-ω and *PI*-λ in comparison to *FI* in Figure 6. Once again, it is evident that observing the realization of ω is more crucial than observing the realization of λ in each of the displayed scenarios. The relative difference between both partial information mechanisms is more pronounced for a state with severe scarcity ($C \leq T$) than for one with moderate scarcity ($C \geq 2 \cdot T$). Moreover, the gap between those two mechanisms is greatest for $\omega \sim U[0,1], \lambda \sim N[0.5,0.1,0,1]$ and smallest for $\omega \sim N[0.5,0.1,0,1], \lambda \sim U[0,1]$. This emphasizes that observing a random variable with a higher deviation carries more potential than observing a random variable with a lower deviation (although it is still not enough for *PI*-λ to surpass *PI*-ω in the latter scenario). Lastly, in states with severe scarcity ($C \leq T$), observing ω is almost as beneficial as observing both ω and λ. This is most noticeable in the third and fourth scenarios where $\omega \sim N[0.5,0.1,0,1]$, there is a high chance of a moderate to high realization of ω. For example, there is roughly a 70% chance of observing a realization of $w \geq 0.45$. Thereby, most of the time, it is favorable to sell at least the first unit in every period. As there is not enough capacity to sell more than one unit on average, a second unit is seldom sold in any period (the price of the second unit is going to be quite high, and thus, a second unit is only sold if the realization of λ is close to 1). For $C = T$, this is most apparent. The expected revenue in every period is close to the expected value of ω (0.5), and accordingly, the expected revenue for $(C,T) = (40,40)$ is close to 20 for *PI*-ω and *FI* (cf. Table 3).

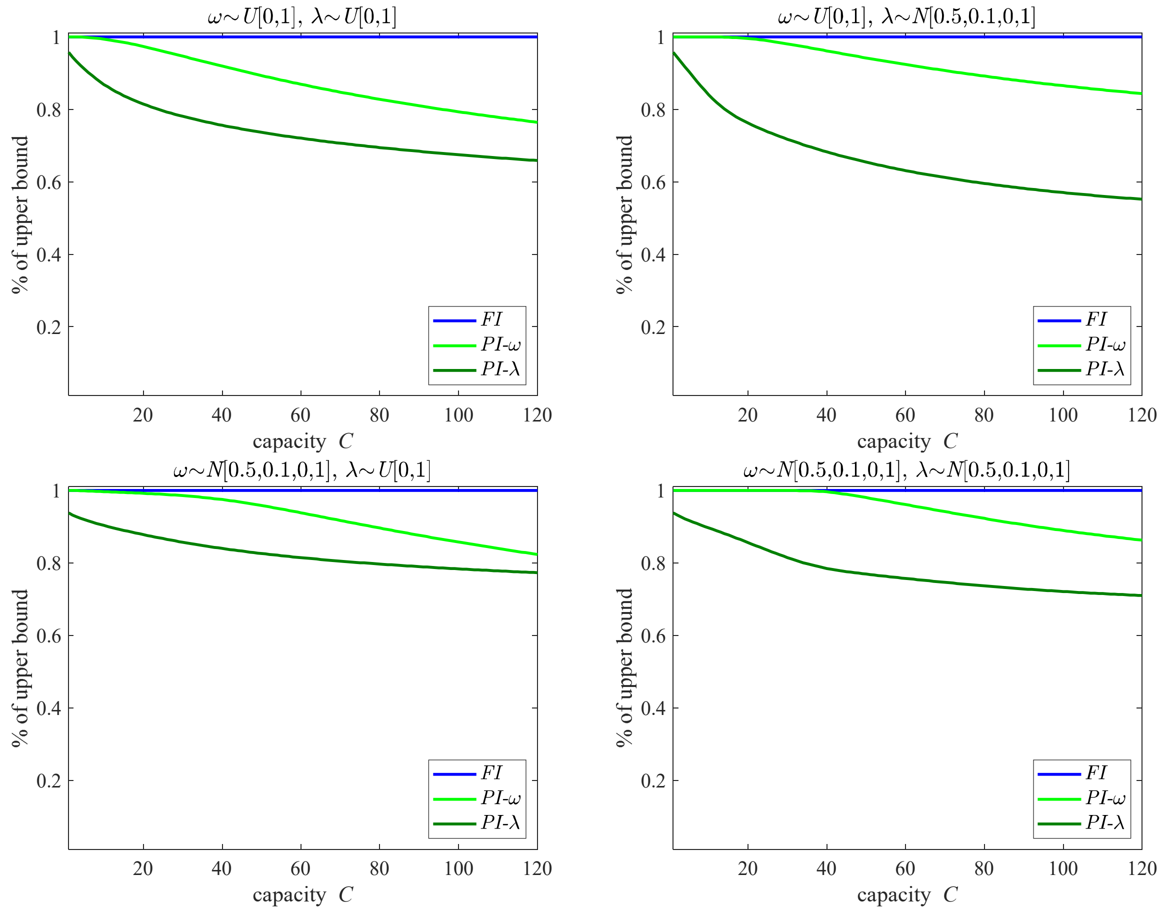

Figure 6: Partially observable information (*PI-ω* and *PI-λ*) relative to full information (*FI*) for $C \leq 120, T = 40$ and different distributions

Finally, we have a closer look at the third scenario, i.e., $\omega \sim N[0.5, 0.1, 0, 1], \lambda \sim U[0,1]$. We have seen that observing λ was almost as good as observing ω for $C = 120$. Indeed, it is evident that observing λ becomes more crucial in states with less scarcity. Scarcity can be described by the ratio T/C, as less time (i.e., demand) or more capacity decreases scarcity. In our simulation study, scarcity varies from $1/120$ to $120/1$. For each $T \leq 40$, we assessed whether *PI-λ* outperforms *PI-ω* for some capacity $C \leq 120$. We found that for $T \leq 27$, there is always a capacity $C^s(T)$ such that *PI-λ* outperforms *PI-ω* for $C \geq C^s(T)$. This $C^s(T)$ forms a line with a slope of approximately 4.5 (cf. Figure 7). It is worth noting that this slope represents the minimum scarcity for which *PI-λ* outperforms *PI-ω*.

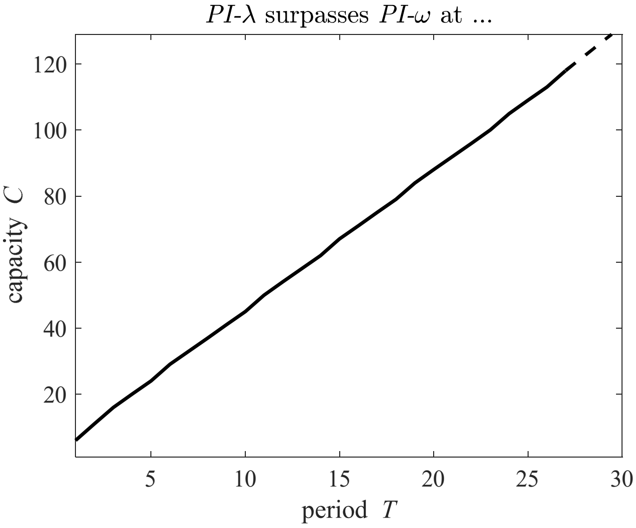

Figure 7: Minimal capacity $C^s(T)$ for which *PI-λ* outperforms *PI-ω*

5.4 Customer segmentation

In this section, we relax our initial assumption that realizations of random variables can be precisely observed. Instead, we consider predefined customer segments and assume the firm can accurately assign arriving customers to these segments. Technically, we divide $[0, 1]$ into several disjunct intervals, and we assume that the firm can only observe the specific interval to which a realization of the random variable belongs.

There are different approaches to designing N intervals $[a_n, b_n]$, where $n \leq N$, with $b_n = a_{n+1}$ for $n \leq N - 1$, $a_1 = 0$, and $b_N = 1$. Note that these intervals are almost surely disjunct, which is sufficient in our setting. One approach could be to employ equidistant intervals, i.e., $b_n - a_n = b_m - a_m$ for all $m, n \leq N$. Another approach is to use equally likely intervals, i.e., $F(b_n) - F(a_n) = F(b_m) - F(a_m)$ for all $m, n \leq N$. Under a uniform distribution, which is employed in this section, both approaches lead to the same intervals. We assume that the firm can observe the correct interval $[a_n, b_n]$ to which the realization of ω (*PI*-ω), λ (*PI*-λ), or both (*FI*) belongs.

The firm then utilizes the (conditional) mean of this interval, calculated as $\frac{1}{F(b_n)-F(a_n)} \int_{a_n}^{b_n} x \cdot f(x)\, dx$, as an estimate for the unknown precise realization. Unobserved parameters, such as ω in *PI*-λ, are treated as random variables. Employing such an estimate transforms our mechanisms (*PI*-ω, *PI*-λ, and *FI*) into heuristics (*H*-ω, *H*-λ, and *H-FI*), resulting in calculated revenues (based on the estimate) that may differ from simulated revenues (based on realizations).

Moreover, we adapted the main idea behind heuristic *D* from Schur (2023) to create another heuristic designed to work with truncated uniform distributions. We made two modifications to the original formulation of *D*: First, the underlying uniform distribution is no longer required to be $U[0, 1]$ but can be truncated on any interval $[a_n, b_n]$, i.e. $U[a_n, b_n]$. Second, we omitted the part involving the fluid approximation as we lacked the necessary analytical results to efficiently solve it for truncated uniform distributions.

We implemented three versions of this heuristic, namely, *D-*ω, *D-*λ, and *D-FI*. For these versions, we assume the observation of the correct interval $[a_n, b_n]$ to which the realization of ω (*D-*ω), λ (*D-*λ), or both (*D-FI*) belongs, and utilize the truncated probability distribution $U[a_n, b_n]$ for the corresponding random variable.

In our simulation study, we examine six scenarios resulting from a combination of three different kinds of observable information (ω, λ, or both) and two different degrees of customer segmentation (size of N). We chose a very low N $(= 2)$ and a medium-sized N $(= 5)$. Apparently, for a large N, we would obtain almost identical results to those presented in Section 5.1. In each scenario, we apply two heuristics, one from our mechanisms with the corresponding estimate (*H-*ω, *H-*λ, and *H-FI)* and one of the three versions of *D* (*D-*ω, *D-*λ, and *D-FI)*.

The findings from our simulation study are presented in Table 4, with each column corresponding to one of the six scenarios and showcasing the revenues generated by the respective *H* and *D* heuristics. A notable observation emerges: consistently, *D* outperforms *H*. This suggests that neglecting uncertainty in observed parameters (by assuming an estimate instead of a random variable on a truncated distribution) is more detrimental than substituting the true customer choice model with a simplified version.

Table 4: Revenues for $C \leq 120, T = 40$ under customer segmentation

	$N = 2$			$N = 5$		
$\boldsymbol{T = 40}$	*H-FI*	*H-*ω	*H-*λ	*H-FI*	*H-*ω	*H-*λ
$\boldsymbol{C = 1}$	0.50 €	0.50 €	0.91 €	0.89 €	0.89 €	0.91 €
$\boldsymbol{C = 20}$	9.61 €	9.93 €	11.58 €	12.75 €	12.78 €	12.19 €
$\boldsymbol{C = 40}$	14.90 €	17.74 €	19.10 €	20.36 €	20.50 €	19.33 €
$\boldsymbol{C = 60}$	17.81 €	23.05 €	23.71 €	25.71 €	26.02 €	24.08 €
$\boldsymbol{C = 80}$	19.77 €	26.90 €	26.62 €	29.56 €	30.14 €	27.43 €
$\boldsymbol{C = 100}$	20.70 €	29.84 €	28.63 €	32.41 €	33.31 €	29.89 €
$\boldsymbol{C = 120}$	21.33 €	32.15 €	30.04 €	34.65 €	35.85 €	31.76 €
$\boldsymbol{T = 40}$	*D-FI*	*D-*ω	*D-*λ	*D-FI*	*D-*ω	*D-*λ
$\boldsymbol{C = 1}$	0.91 €	0.91 €	0.91 €	0.92 €	0.92 €	0.91 €
$\boldsymbol{C = 20}$	12.64 €	12.62 €	12.44 €	13.83 €	13.75 €	12.53 €
$\boldsymbol{C = 40}$	20.26 €	20.16 €	19.44 €	22.84 €	22.35 €	19.73 €
$\boldsymbol{C = 60}$	25.73 €	25.51 €	24.22 €	29.40 €	28.31 €	24.76 €
$\boldsymbol{C = 80}$	29.86 €	29.49 €	27.72 €	34.40 €	32.72 €	28.55 €
$\boldsymbol{C = 100}$	33.07 €	32.57 €	30.43 €	38.36 €	36.12 €	31.53 €
$\boldsymbol{C = 120}$	35.65 €	35.04 €	32.60 €	41.56 €	38.83 €	33.91 €

In D, the order of the value of observable information aligns with the one presented in Section 5.2. However, this is not the case for H. Specifically, H-ω surpasses H-FI and H-λ for $N = 2$ and $N = 5$, and H-λ outperforms H-FI for $N = 2$. The descending order of performance in H-FI provides further evidence of the adverse effects of replacing truncated distributions with estimates, given that two random variables in H-FI are replaced by estimates.

Unsurprisingly, simulated revenues exhibit an upward trend with a more detailed observation of customer segments, applicable to both H and D. To better grasp the impact of the granularity in customer segmentation, a comparison between simulated revenues for D (presented in Table 4) and those of Table 2 is helpful. The simulated revenues for FI, P-ω, and P-λ in Table 2 represent an upper bound, stemming from a scenario where the exact realization was observable, akin to a scenario with $N = \infty$.

The performance of D under $N = 2$ (dashed line) and $N = 5$ (solid line) in relation to their respective upper bounds is visually depicted in Figure 8. It becomes evident that segmenting customers based solely on their consumption indicator (dark green) is notably more robust than segmenting based on their base willingness-to-pay (light green) or a combination of both parameters (blue). However, an ω-based costumer segmentation consistently achieves over 90% of its upper bound. Furthermore, it results in considerably higher simulated revenues than a λ-based customer segmentation (cf. Table 4). This emphasizes the significance of observing ω – the finer the granularity, the better the results.

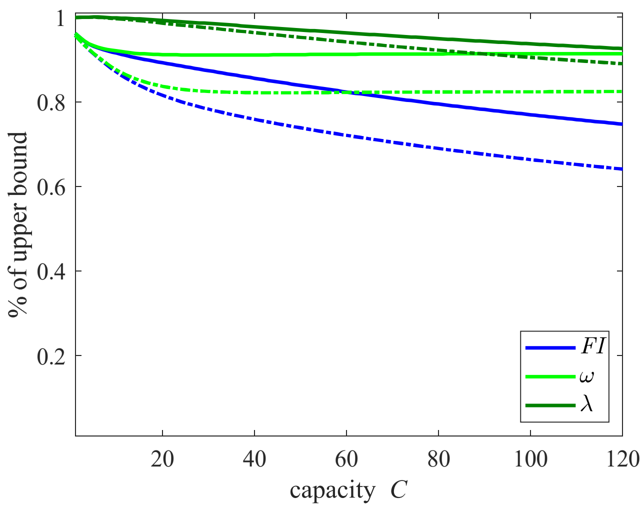

Figure 8: Performance of different customer segmentations with $T = 40$

5.5 Stocking and Restocking

In this section, we introduce an additional layer of decision-making: stocking. Specifically, we consider the firm's ability to determine the initial stocking level. Furthermore, in an extended scenario, we allow the firm to replenish its stock in the middle of the planning horizon at $t = 20$.

For both decisions, we assume that the firm incurs constant unit acquisition costs denoted by s. Therefore, at the beginning of the planning horizon ($T = 40$), the firm must expend $C \cdot s$ to acquire a stock of C units. Consequently, the optimization of the initial decision is expressed as:

$$\max_{C \in \mathbb{Z}} \{V_T(C) - C \cdot s\} \tag{16}$$

Within this maximization framework, $V_T(C)$ can be substituted with $V_T^{\omega,\lambda}(C)$, $V_T^{\omega}(C)$, or $V_T^{\lambda}(C)$, depending on the type of information that is observable.

Additionally, in the restocking scenario the firm has the flexibility to determine the restocking quantity between customers in periods $t = 21$ and $t = 20$. This decision is based on $\max_{x \in \mathbb{Z}} \{V_{20}(c + x) - x \cdot s\}$ with x denoting the restocking quantity. By additionally updating $V_{20}(c + x)$ with $\max_{x \in \mathbb{Z}} \{V_{20}(c + x) - x \cdot s\}$, we proactively incorporate the possibility of restocking between $t = 20$ and $t = 21$ into our pricing decisions for $t \geq 21$. This adaption results in a new optimal policy that leans slightly towards selling more units between $t = 20$ and $t = 40$, as scarcity can be mitigated through the restocking option.

Table 5, displays simulated profits for stocking and restocking scenarios. We examine three distinct acquisition costs ($s \in \{0.3, 0.4, 0.5\}$). However, the integration of a restocking option notably amplifies overall profit, showcasing an improvement of up to 6% (observable ω, $s = 0.5$). Additionally, in the restocking scenario, the optimal initial stock is consistently lower compared to the stocking in a scenario without restocking.

Table 5: Profits for $C \leq 120, T = 40$ under stocking and restocking

	With restocking			Without restocking		
$s = 0.3$	*FI*	*PI-ω*	*PI-λ*	*FI*	*PI-ω*	*PI-λ*
Optimal starting stock C	92	51	41	103	54	44
Simulated profit	20.29 €	13.81 €	8.59 €	19.85 €	13.10 €	8.21 €
$s = 0.4$	*FI*	*PI-ω*	*PI-λ*	*FI*	*PI-ω*	*PI-λ*
Optimal starting stock C	57	35	25	63	38	26
Simulated profit	12.20 €	9.05 €	5.03 €	11.72 €	8.55 €	4.79 €
$s = 0.5$	*FI*	*PI-ω*	*PI-λ*	*FI*	*PI-ω*	*PI-λ*
Optimal starting stock C	36	25	15	39	27	16
Simulated profit	7.08 €	5.68 €	2.85 €	6.70 €	5.35 €	2.70 €

6 Conclusion

In this study, we delved into a dynamic pricing framework encompassing multiunit demands, driven by customers' base willingness-to-pay and consumption indicator. Our exploration considered three scenarios, each involving the firm's observation of the current customer's base willingness-to-pay, consumption indicator, or both. We found the optimality condition for each case. For the second (under uniform distribution) and third case, we derived a closed-form expression of the optimal batch prices.

In contrast to standard singleunit dynamic pricing with time-homogenous demand, economically selling is not always possible in our multiunit dynamic pricing context. In particular, larger batches were frequently priced-out, as convex increasing opportunity costs tended to surpass concave increasing willingness-to-pay. This stands in contrast to singleunit dynamic pricing, where there always exists a price at which the firm can increase its overall expected revenue

We showed well-known monotonicity in time and capacity holds for all cases, inducing an intuitive structure with regard to scarcity of the product, and ensuring the existence of a unique optimal solution. By solving all cases to optimality, we calculated the value of all three types of information a firm might obtain from profiling its current customer. Additionally, we analyzed the impact of customer segmentation when precise observation of customers' private information is unattainable. With this knowledge, a firm gains the ability to assess the profitability of potential investments in customer profiling and segmentation. Furthermore, we provide guidance on leveraging our results to make informed decisions regarding optimal initial stocking and restocking strategies.

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Declarations

The author has no competing interests to declare that are relevant to the content of this article.

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Supplement: Multiunit Dynamic Pricing with Different Types of Observable Customer Information

S.1 Proof of Proposition 3

The value function is obviously increasing in c as there is no downside in having more capacity. Moreover, there is a positive probability to earn additional revenue making it strictly increasing for $t \geq 1$. We show the concavity of $V_t^\omega(c|w) = V_{t-1}^\omega(c) + \sum_{j=1}^{c} \max_{\Delta r_j \in [0,w]} \left\{ \left(1 - F_\lambda\left(\underline{l}_j(\Delta r_j)\right)\right) \cdot \left(\Delta r_j - \Delta_1 V_{t-1}^\omega(c+1-j)\right) \right\} = \sum_{j=1}^{c} m_j\left(\Delta_1 V_{t-1}^\omega(c+1-j)\right)$ by induction over t for any realization w:

<u>$t = 1$:</u> By definition $V_0^\omega(c) = 0$, and thus, $\Delta_1 V_0^\omega(c) = 0$ for every c. We observe

$$\begin{aligned}\Delta_1 V_1^\omega(c|w) = V_1^\omega(c|w) - V_1^\omega(c-1|w) = \sum_{j=1}^{c} m_j(0) - \sum_{j=1}^{c-1} m_j(0) = m_c(0) \geq m_{c+1}(0) \\ = \Delta_1 V_1^\omega(c+1|w).\end{aligned}$$

With Lemma 2 a), it holds that $\Delta_1 V_t^\omega(c|w)$ is decreasing in c, and thus, $V_t^\omega(\cdot\,|w)$ is concave. Finally, it also holds that $V_t^\omega(\cdot) = \int_0^1 V_t^\omega(\cdot\,|w) \cdot f_\omega(w)\, dw$ is concave.

<u>$t \curvearrowright t+1$:</u>

$$
\begin{aligned}
&\Delta_1 V_{t+1}^{\omega}(c+1|w) - \Delta_1 V_{t+1}^{\omega}(c|w) = V_{t+1}^{\omega}(c+1|w) - 2\cdot V_{t+1}^{\omega}(c|w) + V_{t+1}^{\omega}(c-1|w) \\
&\quad = V_t^{\omega}(c) - 2\cdot V_t^{\omega}(c-1) + V_t^{\omega}(c-2) + \sum_{j=2}^{N_{t+1,c+1}(w)} m_j\big(\Delta_1 V_t^{\omega}(c+2-j)\big) - 2 \\
&\quad \cdot \sum_{j=2}^{N_{t+1,c}(w)} m_j\big(\Delta_1 V_t^{\omega}(c+1-j)\big) + \sum_{j=2}^{N_{t+1,c-1}(w)} m_j\big(\Delta_1 V_t^{\omega}(c-j)\big) \\
&\quad = \Delta_1 V_t^{\omega}(c) - \Delta_1 V_t^{\omega}(c-1) + m_2\big(\Delta_1 V_t^{\omega}(c)\big) - m_2\big(\Delta_1 V_t^{\omega}(c-1)\big) \\
&\quad + m_{N_{t+1,c+1}(w)}\Big(\Delta_1 V_t^{\omega}\Big(c+2-N_{t+1,c+1}(w)\Big)\Big) \\
&\quad - m_{N_{t+1,c}(w)}\Big(\Delta_1 V_t^{\omega}\Big(c+1-N_{t+1,c}(w)\Big)\Big) \\
&\quad + \sum_{j=2}^{N_{t+1,c-1}(w)} \Big(\Big(m_{j+1}\big(\Delta_1 V_t^{\omega}(c+1-j)\big) - m_j\big(\Delta_1 V_t^{\omega}(c+1-j)\big)\Big) \\
&\quad - \Big(m_{j+1}\big(\Delta_1 V_t^{\omega}(c-j)\big) - m_j\big(\Delta_1 V_t^{\omega}(c-j)\big)\Big)\Big) \\
&\quad \le \Delta_1 V_t^{\omega}(c) - \Delta_1 V_t^{\omega}(c-1) + m_2\big(\Delta_1 V_t^{\omega}(c)\big) - m_2\big(\Delta_1 V_t^{\omega}(c-1)\big) \\
&\quad + m_{N_{t+1,c+1}(w)}\Big(\Delta_1 V_t^{\omega}\Big(c+2-N_{t+1,c+1}(w)\Big)\Big) \\
&\quad - m_{N_{t+1,c}(w)}\Big(\Delta_1 V_t^{\omega}\Big(c+1-N_{t+1,c}(w)\Big)\Big) \\
&\quad \le \Delta_1 V_t^{\omega}(c) - \Delta_1 V_t^{\omega}(c-1) + m_2\big(\Delta_1 V_t^{\omega}(c)\big) - m_2\big(\Delta_1 V_t^{\omega}(c-1)\big) \\
&\quad \le \Delta_1 V_t^{\omega}(c) - \Delta_1 V_t^{\omega}(c-1) - \Big(1 - F_\lambda\Big(\underline{l}_2\big(\Delta_1 V_t^{\omega}(c)\big)\Big)\Big)\cdot\big(\Delta_1 V_t^{\omega}(c) - \Delta_1 V_t^{\omega}(c-1)\big) \\
&\quad \le 0
\end{aligned}
$$

The first and second inequality follow by Lemma 2 and the induction hypothesis, the third inequality by $m_2\big(\Delta_1 V_t^{\omega}(c-1)\big) \ge \Big(1 - F_\lambda\Big(\underline{l}_2\big(\Delta_1 V_t^{\omega}(c)\big)\Big)\Big)\cdot\Big(w\cdot \underline{l}_2\big(\Delta_1 V_t^{\omega}(c)\big) - \Delta_1 V_t^{\omega}(c-1)\Big)$ with $\underline{l}_2\big(\Delta_1 V_t^{\omega}(c)\big)$ being the optimal solution to $m_2\big(\Delta_1 V_t^{\omega}(c)\big)$. Finally, it holds that

$$\Delta_1 V_{t+1}^{\omega}(c+1) - \Delta_1 V_{t+1}^{\omega}(c) = \int_0^1 \big(\Delta_1 V_{t+1}^{\omega}(c+1|w) - \Delta_1 V_{t+1}^{\omega}(c|w)\big)\cdot f_\omega(w)\,dw \le 0.$$

□

S.2 Proof of Proposition 4

We show part a), i.e. $\Delta_1 V_t^{\omega}(c) \ge \Delta_1 V_{t-1}^{\omega}(c)$, by induction:

<u>$t = 1$:</u>

$$\Delta_1 V_1^{\omega}(c|w) - \Delta_1 V_0^{\omega}(c|w) = \Delta_1 V_1^{\omega}(c|w) \ge 0$$

$\underline{t \curvearrowright t+1:}$

$$\begin{aligned}
\Delta_1 V_{t+1}^{\omega}(c|w) - \Delta_1 V_t^{\omega}(c|w) &= V_{t+1}^{\omega}(c|w) - V_{t+1}^{\omega}(c-1|w) - V_t^{\omega}(c|w) + V_t^{\omega}(c-1|w) \\
&= V_t^{\omega}(c-1) - V_t^{\omega}(c-2) - V_{t-1}^{\omega}(c-1) + V_{t-1}^{\omega}(c-2) \\
&+ \sum_{j=2}^{N_{t+1,c}(w)} m_j\big(\Delta_1 V_t^{\omega}(c+1-j)\big) - \sum_{j=2}^{N_{t+1,c-1}(w)} m_j\big(\Delta_1 V_t^{\omega}(c-j)\big) \\
&- \sum_{j=2}^{N_{t,c}(w)} m_j\big(\Delta_1 V_{t-1}^{\omega}(c+1-j)\big) + \sum_{j=2}^{N_{t,c-1}(w)} m_j\big(\Delta_1 V_{t-1}^{\omega}(c-j)\big) \\
&= \Delta_1 V_t^{\omega}(c-1) - \Delta_1 V_{t-1}^{\omega}(c-1) + m_2\big(\Delta_1 V_t^{\omega}(c-1)\big) - m_2\big(\Delta_1 V_{t-1}^{\omega}(c-1)\big) \\
&+ \sum_{j=2}^{N_{t+1,c-1}(w)} \Big(\Big(m_{j+1}\big(\Delta_1 V_t^{\omega}(c-j)\big) - m_j\big(\Delta_1 V_t^{\omega}(c-j)\big)\Big) \\
&- \Big(m_{j+1}\big(\Delta_1 V_{t-1}^{\omega}(c-j)\big) - m_j\big(\Delta_1 V_{t-1}^{\omega}(c-j)\big)\Big)\Big) - 1_{\{N_{t,c-1}(w) \geq N_{t+1,c-1}(w)+1\}} \\
&\cdot \sum_{j=N_{t+1,c-1}(w)+1}^{N_{t,c-1}(w)} \Big(m_{j+1}\big(\Delta_1 V_{t-1}^{\omega}(c-j)\big) - m_j\big(\Delta_1 V_{t-1}^{\omega}(c-j)\big)\Big) \\
&\geq \Delta_1 V_t^{\omega}(c-1) - \Delta_1 V_{t-1}^{\omega}(c-1) + m_2\big(\Delta_1 V_t^{\omega}(c-1)\big) - m_2\big(\Delta_1 V_{t-1}^{\omega}(c-1)\big) \\
&\geq \Delta_1 V_t^{\omega}(c-1) - \Delta_1 V_{t-1}^{\omega}(c-1) - \Big(1 - F_\lambda\Big(\underline{l}_2\big(\Delta_1 V_{t-1}^{\omega}(c-1)\big)\Big)\Big) \\
&\cdot \big(\Delta_1 V_t^{\omega}(c-1) - \Delta_1 V_{t-1}^{\omega}(c-1)\big) \geq 0
\end{aligned}$$

with $N_{t+1,c}(w) = N_{t+1,c-1}(w) + 1$, and $N_{t,c}(w) = N_{t,c-1}(w) + 1$. Moreover, it holds that $N_{t+1,c-1}(w) \leq N_{t,c-1}(w)$. The inequalities follow by Lemma 2, the induction hypothesis, $m_{j+1}(\delta) \leq m_j(\delta)$, and suboptimality of $\underline{l}_2\big(\Delta_1 V_{t-1}^{\omega}(c-1)\big)$ for $m_2\big(\Delta_1 V_t^{\omega}(c-1)\big)$. The last step is given by $\Delta_1 V_{t+1}^{\omega}(c) - \Delta_1 V_t^{\omega}(c) = \int_0^1 \Delta_1 V_{t+1}^{\omega}(c|w) - \Delta_1 V_t^{\omega}(c|w) \cdot f_\omega(w)\, dw \geq 0$.

b) There is no downside in having more time to sell remaining capacity. In the contrary, there is always a positive probability for earning additional revenues by having additional selling opportunities. Thus, value function $V_t^{\omega}(c)$ is increasing. Concavity, i.e. $V_{t+1}^{\omega}(c) - V_t^{\omega}(c) \leq V_t^{\omega}(c) - V_{t-1}^{\omega}(c)$ is given:

$$\begin{aligned}
&V_{t+1}^{\omega}(c|w) - V_t^{\omega}(c) - V_t^{\omega}(c|w) + V_{t-1}^{\omega}(c) \\
&\quad = \Delta_1 V_{t-1}^{\omega}(c) - \Delta_1 V_t^{\omega}(c) \\
&\quad + \sum_{j=2}^{N_{t+1,c}(w)} \Big(m_j\big(\Delta_1 V_t^{\omega}(c+1-j)\big) - m_j\big(\Delta_1 V_{t-1}^{\omega}(c+1-j)\big)\Big) - 1_{\{N_{t,c}(w) \geq N_{t+1,c}(w)+1\}} \\
&\quad \cdot \sum_{j=N_{t+1,c}(w)+1}^{N_{t,c}(w)} m_j\big(\Delta_1 V_{t-1}^{\omega}(c+1-j)\big) \leq 0
\end{aligned}$$

with $N_{t,c}(w) \geq N_{t+1,c}(w)$ as $\Delta_1 V_t^{\omega}(c+1-j) \leq \Delta_1 V_{t+1}^{\omega}(c+1-j)$ (cf. part a) of this proposition). Inequality follows by Lemma 2 and again part a) of this proposition. Finally, $V_{t+1}^{\omega}(c) - 2V_t^{\omega}(c) + V_{t-1}^{\omega}(c-1) = \int_0^1 \left(V_{t+1}^{\omega}(c|w) - V_t^{\omega}(c) - V_t^{\omega}(c|w) + V_{t-1}^{\omega}(c)\right) \cdot f_{\omega}(w)\, dw \leq 0.$ □

S.3 Proof of Proposition 5

Proof of Proposition 5 consists of similar steps as the proofs of Propositions 1 and 2 with slightly different math, stemming from our knowledge regarding customer's consumption indicator (instead of customer's base willingness-to-pay). Thus, we follow the same roadmap:

1. Break down the optimization problem into $N_{t,c}(l)$ separate optimization problems that are not connected to each other
2. Show that the first order condition is sufficient to find the unique optimal solution to each of these problems by:
    a. Replacing Δr_j by $\theta = \left(1 - F_{\omega}\left(\frac{\Delta r_j}{l^{j-1}}\right)\right)$ as our decision variable to eliminate opportunity costs in our second derivative
    b. Calculating the first derivative to have a necessary condition on optimality and showing the existence of an optimal solution on $(0,1)$
    c. Calculating the second derivative to prove concavity of the optimization problem
3. Show that the optimal solutions of the $N_{t,c}(l)$ separate optimization problems constitute the optimal solution of our original optimization problem

1. We build the related optimization problems, $j = 1, \ldots, N_{t,c}(l)$:

$$\max_{\Delta r_j \in [0, l^{j-1}]} \left\{\left(1 - F_{\omega}\left(\frac{\Delta r_j}{l^{j-1}}\right)\right) \cdot \left(\Delta r_j - \Delta_1 V_{t-1}^{\lambda}(c+1-j)\right)\right\}$$

The idea behind this formulation is to optimize marginal prices separately. Thereby, we technically assume that customers are allowed to separately purchase the jth unit without purchasing the 1st, 2nd, …, or $j-1$th unit of the product. On a more formal note: We drop the conditions on prices that are formulated in $\mathcal{R}_c(l)$. Thereby, we created $N_{t,c}(l)$ optimization problems whose sum is an upper bound to our original problem. We want to stress that the objective value of our original problem for some price vector $\boldsymbol{r} \in \mathcal{R}_c(l)$ equals the sum of the objective values of the problems defined above for $\Delta r_j = r_j - r_{j-1}$. This follows by definition of $p_j(\boldsymbol{r}|l)$.

2.a. We define $\theta = \left(1 - F_\omega\left(\frac{\Delta r_j}{l^{j-1}}\right)\right)$ to rewrite the optimization problem for every j:

$$\max_{\Delta r_j \in [0, l^{j-1}]} \left\{\left(1 - F_\omega\left(\frac{\Delta r_j}{l^{j-1}}\right)\right) \cdot \left(\Delta r_j - \Delta_1 V_{t-1}^{\lambda}(c+1-j)\right)\right\} = \max_{\theta \in [0,1]} \left\{\theta \cdot \left(l^{j-1} \cdot F_\omega^{-1}(1-\theta) - \Delta_1 V_{t-1}^{\lambda}(c+1-j)\right)\right\}$$

with F_ω^{-1} being the inverse of F_ω. Moreover, we want to stress that θ and Δr_j are connected by a bijective function. Hence, there is a unique matching between these two variables.

2.b. Building the first derivative, we get

$$\frac{d}{d\theta}\theta \cdot \left(l^{j-1} \cdot F_\omega^{-1}(1-\theta) - \Delta_1 V_{t-1}^{\lambda}(c+1-j)\right)$$
$$= l^{j-1} \cdot F_\omega^{-1}(1-\theta) - \Delta_1 V_{t-1}^{\lambda}(c+1-j) - \theta \cdot l^{j-1} \cdot \frac{1}{f_\omega\left(F_\omega^{-1}(1-\theta)\right)}.$$

On the boundaries, the first derivative is positive for $\theta = 0$ (remember that $l^{j-1} > \Delta_1 V_{t-1}^{\lambda}(c+1-j)$ and $f_\omega > 0$) and negative for $\theta = 1$. Together with continuity of the first derivative, there must be at least one $\theta \in (0,1)$ that meets the first order condition. By defining $\underline{w}_j = F_\omega^{-1}(1-\theta)$ (or equivalently, $\underline{w}_j = \frac{\Delta r_j}{l^{j-1}}$), we can reformulate the first order condition to

$$l^{j-1} \cdot \left(\underline{w}_j - \frac{1}{h_\omega\left(\underline{w}_j\right)}\right) = \Delta_1 V_{t-1}^{\lambda}(c+1-j)$$

which is the formulation stated in Proposition 5.

2.c. To calculate the second derivative, we use the reformulation above. Moreover, we write $\underline{w}_j(\theta)$ to highlight that θ is the decision variable and $\underline{w}_j$ depends on θ.

$$\frac{d^2}{d\theta^2}\theta \cdot \left(l^{j-1} \cdot F_\omega^{-1}(1-\theta) - \Delta_1 V_{t-1}^\lambda(c+1-j)\right)$$

$$= \frac{d}{d\theta}\, l^{j-1} \cdot \left(\underline{w}_j(\theta) - \frac{1}{h_\omega\left(\underline{w}_j(\theta)\right)} \right) - \Delta_1 V_{t-1}^\lambda(c+1-j)$$

$$= l^{j-1} \cdot \left(1 + \frac{h_\omega{}'\left(\underline{w}_j(\theta)\right)}{\left(h_\omega\left(\underline{w}_j(\theta)\right)\right)^2} \right) \cdot \frac{d}{d\theta}\underline{w}_j(\theta)$$

The second derivative is negative as the failure rate h_ω is increasing and $\underline{w}_j(\theta)$ is strictly decreasing in θ. Consequently, the optimization problem is strictly concave. Together with our proof of existence (2.b.), we have shown that there is exactly one solution to our optimization model and it is well-defined by the first order condition.

3. The remaining step of our proof is to show that the optimal solutions $\Delta r_{t,j}(c|l)$ to our $N_{t,c}(l)$ optimization problems constitute the unique optimal solution of our original optimization problem. This can be done by showing that price vector $\boldsymbol{r}_t(c|l)$ with $r_{t,j}(c|l) = \sum_{i=1}^{j} \Delta r_{t,i}(c|l)$ is element of $\mathcal{R}_c(l)$ and actually has to be the unique optimal solution.

For $r_{t,j}(c|l) = \sum_{i=1}^{j} \Delta r_{t,i}(c|l) \in \mathcal{R}_c(l)$, we merely have to show that $\frac{\Delta r_{t,j}(c|l)}{l^{j-1}} \leq \frac{\Delta r_{t,j+1}(c|l)}{l^j}$ for every $j < N_{t,c}(l)$. This is equivalent to showing $\underline{w}_j \leq \underline{w}_{j+1}$, $j < N_{t,c}(l)$, with the definition from 2.b. Recalling the optimality condition, it holds that

$$0 = l^{j-1} \cdot \left(\underline{w}_j - \frac{1}{h_\omega(\underline{w}_j)} \right) - \Delta_1 V_{t-1}^\lambda(c+1-j) \geq l^j \cdot \left(\underline{w}_j - \frac{1}{h_\lambda(\underline{w}_j)} \right) - \Delta_1 V_{t-1}^\lambda(c+1-j)$$

$$\geq l^j \cdot \left(\underline{w}_j - \frac{1}{h_\omega(\underline{w}_j)} \right) - \Delta_1 V_{t-1}^\lambda(c-j).$$

The first inequality follows by $l \in [0,1]$ and $\underline{w}_j - \frac{1}{h_\omega(\underline{w}_j)} \geq 0$, the second by concavity of $V_{t-1}^\lambda(c)$ in c. So far, we have shown that the first derivative for $j+1$ is nonpositive at $\underline{w}_j$. As the first derivative increases in $\underline{w}$, it must hold that $\underline{w}_{j+1} \geq \underline{w}_j$ for some $\underline{w}_{j+1}$ that meets the optimality condition for $j+1$.

The uniqueness shown in 2.c. carries over to the solution of our original problem. This can be proven by contradiction: We assume that there is another price vector $\boldsymbol{r} \in \mathcal{R}_c(l)$ that leads to an equally good or better solution of our original problem Then we can disassemble this price vector in $N_{t,c}(l)$ marginal prices $\Delta r_j = r_j - r_{j-1}$. Together with the argumentation presented in 1., this would imply the sum of the objective functions would be equally good or better than the sum of the optimal objective values. This would be a contradiction to the uniqueness or the optimality, respectively, of the optimal solution of at least one of these optimization problems.

S.4 Definition of Lemma S1 and its proof

We can confirm that the statements regarding the expected additional revenue for selling the $j+1$th unit we established in Lemma 2 (Section 4.1) also hold in Section 4.2 with observable consumption indicator and unobservable base willingness-to-pay. Analogue to Section 4.1, we define $m_j(\delta) = \max_{\Delta r_j \in [0, l^{j-1}]} \left\{ \left(1 - F_\omega\left(\frac{\Delta r_j}{l^{j-1}}\right)\right) \cdot \left(\Delta r_j - \Delta_1 V_{t-1}^\lambda(c+1-j)\right) \right\}$ and state

Lemma S1 *If* $\delta \in \left[0, l^{j-1}\right)$*, it holds that:*

a) $m_{j+1}(\delta) - m_j(\delta) \leq 0$

b) $m_{j+1}(\delta) - m_j(\delta)$ *is increasing in* δ

c) $m_j(\delta)$ *is decreasing in* δ

Proof: This proof follows the exact same steps and the same intentions as proof of Lemma 2. The only thing that changes is the math.

a): $m_j(\delta)$ is the optimal value of $\max_{\Delta r_j \in [0, l^{j-1}]} \left\{ \left(1 - F_\omega\left(\frac{\Delta r_j}{l^{j-1}}\right)\right) \cdot \left(\Delta r_j - \Delta_1 V_{t-1}^\lambda(c+1-j)\right) \right\} = \max_{\underline{w}_j \in [0,1]} \left\{ \left(1 - F_\omega(\underline{w}_j)\right) \cdot \left(\underline{w}_j \cdot (l)^{j-1} - \delta\right) \right\} = \left(1 - F_\omega\left(\underline{w}_j(l)\right)\right) \cdot \left(\underline{w}_j(l) \cdot l^{j-1} - \delta\right)$ with $\underline{w}_j(l)$ representing the optimal solution. As $\underline{w}_{j+1}(l)$ (the optimal solution of $m_{j+1}(\delta)$) is suboptimal for $m_j(\delta)$ and $1 - F_\omega\left(\underline{w}_{j+1}(l)\right) \geq 0$ as well as $\underline{w}_{j+1}(l) \leq 1$, it holds that

$$m_j(\delta) = \max_{\underline{w}_j \in [0,1]} \left\{ \left(1 - F_\omega(\underline{w}_j)\right) \cdot \left(\underline{w}_j \cdot (l)^{j-1} - \delta\right) \right\} \geq \left(1 - F_\omega\left(\underline{w}_{j+1}(l)\right)\right) \cdot \left(\underline{w}_{j+1}(l) \cdot l^{j-1} - \delta\right)$$
$$\geq \left(1 - F_\omega\left(\underline{w}_{j+1}(l)\right)\right) \cdot \left(\underline{w}_{j+1}(l) \cdot l^j - \delta\right) = m_{j+1}(\delta)$$

b) and c): In a first step, we prove c) by formulating the first derivative of $m_j(\delta)$ with regard to δ and observing its nonpositivity. In a second step we prove b) by applying the formulation of our first step.

Based on its implicit definition $l^{j-1} \cdot \left(\underline{w}_j - \frac{1}{h_\omega(\underline{w}_j)}\right) = \delta$ (cf. Proposition 5), the optimal solution $\underline{w}_j(l)$ of $m_j(\delta)$ depends also on δ. As we are about to vary δ, we highlight this fact by writing $\underline{w}_j(\delta)$ instead of $\underline{w}_j(l)$ (l acts as a parameter in this proof). The same applies for $\underline{w}_{j+1}(\delta)$ and $m_{j+1}(\delta)$. Building the first derivative, we get

$$\begin{aligned}\frac{d}{d\delta} m_j(\delta) &= \frac{d}{d\delta}\left(\left(1 - F_\omega\left(\underline{w}_j(\delta)\right)\right) \cdot \left(\underline{w}_j(\delta) \cdot l^{j-1} - \delta\right)\right) \\ &= -f_\omega\left(\underline{w}_j(\delta)\right) \cdot \frac{d}{d\delta}\left(\underline{w}_j(\delta)\right) \cdot \left(\underline{w}_j(\delta) \cdot l^{j-1} - \delta\right) + \left(1 - F_\omega\left(\underline{w}_j(\delta)\right)\right) \\ &\quad \cdot \left(l^{j-1} \cdot \frac{d}{d\delta}\left(\underline{w}_j\right) - 1\right) \\ &= \frac{d}{d\delta}\left(\underline{w}_j(\delta)\right) \cdot f_\omega\left(\underline{w}_j(\delta)\right) \cdot \left(l^{j-1} \cdot \left(\frac{1}{h_\omega\left(\underline{w}_j(\delta)\right)} - \underline{w}_j(\delta)\right) + \delta\right) \\ &\quad - \left(1 - F_\omega\left(\underline{w}_j(\delta)\right)\right) = -\left(1 - F_\omega\left(\underline{w}_j(\delta)\right)\right) \leq 0.\end{aligned}$$

The last equation holds because of the implicit definition of $\underline{w}_j(\delta)$.

After showing c), we make use of the formulation above. Please note, that replacing j by $j+1$ does not change the argumentation. It immediately follows that $\frac{d}{d\delta} m_{j+1}(\delta) = -\left(1 - F_\omega\left(\underline{w}_j(\delta)\right)\right)$. Combining the first derivative of $m_j(\delta)$ and $m_{j+1}(\delta)$ leads to

$$\frac{d}{d\delta}\left(m_{j+1}(\delta) - m_j(\delta)\right) = F_\omega\left(\underline{w}_{j+1}(\delta)\right) - F_\omega\left(\underline{w}_j(\delta)\right).$$

Recalling the argumentation while developing Proposition 5, we know that $\underline{w}_{j+1}(\delta) \geq \underline{w}_j(\delta)$. Hence, we can conclude that $\frac{d}{d\delta}\left(m_{j+1}(\delta) - m_j(\delta)\right) \geq 0$. □

Remark S1 *By definition, it holds that* $m_j\left(l^{j-1}\right) = 0$ *with* $\underline{w}_j\left(l^{j-1}\right) = 1$. *Although not exactly stated, we assumed* $\delta < l^{j-1}$ *in proof of Lemma S1. However, all statements also hold for* $\delta \geq l^{j-1}$.

S.5 Proof of Proposition 6

$V_t^\lambda(\cdot)$ is obviously increasing as it is never harmful to have more capacity. We show concavity by induction:

$t = 1$:

With Lemma S1, it holds

$$\Delta_1 V_1^{\lambda}(c+1|l) - \Delta_1 V_1{}^{\lambda}(c|l) = m_{c+1}(0) - m_c(0) \leq 0.$$

Subsequently, it also holds

$$\Delta_1 V_1^{\lambda}(c+1) - \Delta_1 V_1^{\lambda}(c) = \int_0^1 \Delta_1 V_1^{\lambda}(c+1|l) - \Delta_1 V_1{}^{\lambda}(c|l) \cdot f_{\lambda}(l)\, dl \leq 0$$

<u>$t \curvearrowright t+1$:</u>

In the induction step, we first show that $\Delta_1 V_{t+1}^{\lambda}(c+1|l) - \Delta_1 V_{t+1}^{\lambda}(c|l) \leq 0$ and conclude that $\Delta_1 V_{t+1}^{\lambda}(c+1) - \Delta_1 V_{t+1}^{\lambda}(c) \leq 0$ holds as well. We also recall Remark 4 which says that $N_{t+1,c-1}(l) \leq N_{t+1,c}(l) \leq N_{t+1,c-1}(l) + 1$. This translates to the following four cases (with $N = N_{t+1,c-1}(l)$ to shorten notation):

a) $N = N_{t+1,c}(l) = N_{t+1,c+1}(l)$

b) $N = N_{t+1,c}(l) = N_{t+1,c+1}(l) - 1$

c) $N = N_{t+1,c}(l) - 1 = N_{t+1,c+1}(l) - 1$

d) $N = N_{t+1,c}(l) - 1 = N_{t+1,c+1}(l) - 2$

$$\Delta_1 V_{t+1}^{\lambda}(c+1|l) - \Delta_1 V_{t+1}^{\lambda}(c|l) = V_{t+1}^{\lambda}(c+1|l) - 2V_{t+1}^{\lambda}(c|l) + V_{t+1}^{\lambda}(c-1|l)$$

$$= V_t^{\lambda}(c+1|l) - 2V_t{}^{\lambda}(c|l) + V_t{}^{\lambda}(c-1|l) + \sum_{j=2}^{N_{t+1,c+1}(l)} m_j\left(\Delta_1 V_t^{\lambda}(c+2-j)\right) - 2$$

$$\cdot \sum_{j=2}^{N_{t+1,c}(l)} m_j\left(\Delta_1 V_t^{\lambda}(c+1-j)\right) + \sum_{j=2}^{N} m_j\left(\Delta_1 V_t^{\lambda}(c-j)\right)$$

$$= \Delta_1 V_t^{\lambda}(c) - \Delta_1 V_t^{\lambda}(c-1) + m_2\left(\Delta_1 V_t^{\lambda}(c)\right) - m_2\left(\Delta_1 V_t^{\lambda}(c-1)\right)$$

$$+ m_{N+2}\left(\Delta_1 V_t^{\lambda}(c-N)\right) \cdot 1_{\{N=N_{t+1,c+1}(l)-2\}} - m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right)$$

$$\cdot 1_{\{N=N_{t+1,c}(l)-1\}} + m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right) \cdot 1_{\{N=N_{t+1,c}(l)\}}$$

$$- m_{N+1}\left(\Delta_1 V_t^{\lambda}(c+1-N)\right) \cdot 1_{\{N=N_{t+1,c+1}(l)\}}$$

$$+ \sum_{j=2}^{N-1}\left(\left(m_{j+1}\left(\Delta_1 V_t^{\lambda}(c+1-j)\right) - m_j\left(\Delta_1 V_t^{\lambda}(c+1-j)\right)\right)\right.$$

$$\left. - \left(m_{j+1}\left(\Delta_1 V_t^{\lambda}(c-j)\right) - m_j\left(\Delta_1 V_t^{\lambda}(c-j)\right)\right)\right)$$

$$+ \left(\left(m_{N+1}\left(\Delta_1 V_t^{\lambda}(c+1-N)\right) - m_N\left(\Delta_1 V_t^{\lambda}(c+1-N)\right)\right)\right.$$

$$\left. - \left(m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right) - m_N\left(\Delta_1 V_t^{\lambda}(c-N)\right)\right)\right)$$

With Lemma S1 b), it holds that

$$\Delta_1 V_{t+1}^{\lambda}(c+1|l) - \Delta_1 V_{t+1}^{\lambda}(c|l)$$

$$\leq \Delta_1 V_t^{\lambda}(c) - \Delta_1 V_t^{\lambda}(c-1) + m_2\left(\Delta_1 V_t^{\lambda}(c)\right) - m_2\left(\Delta_1 V_t^{\lambda}(c-1)\right)$$

$$+ \left(\left(m_{N+1}\left(\Delta_1 V_t^{\lambda}(c+1-N)\right) - m_N\left(\Delta_1 V_t^{\lambda}(c+1-N)\right)\right)\right.$$

$$\left. - \left(m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right) - m_N\left(\Delta_1 V_t^{\lambda}(c-N)\right)\right)\right) + m_{N+2}\left(\Delta_1 V_t^{\lambda}(c-N)\right)$$

$$\cdot 1_{\{N=N_{t+1,c+1}(l)-2\}} - m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right) \cdot 1_{\{N=N_{t+1,c}(l)-1\}}$$

$$+ m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right) \cdot 1_{\{N=N_{t+1,c}(l)\}} - m_{N+1}\left(\Delta_1 V_t^{\lambda}(c+1-N)\right) \cdot 1_{\{N=N_{t+1,c+1}(l)\}}$$

To proof $\Delta_1 V_{t+1}^{\lambda}(c+1|l) - \Delta_1 V_{t+1}^{\lambda}(c|l) \leq 0$, we look at aforementioned cases a) – d). In each of the four cases, the first step is to replace the characteristic functions $1_{\{\cdot\}}$ by 0 and 1.

a) $N = N_{t+1,c}(l) = N_{t+1,c+1}(l)$:

$$\Delta_1 V_{t+1}^{\lambda}(c+1|l) - \Delta_1 V_{t+1}^{\lambda}(c|l)$$

$$\leq \Delta_1 V_t^{\lambda}(c) - \Delta_1 V_t^{\lambda}(c-1) + m_2\left(\Delta_1 V_t^{\lambda}(c)\right) - m_2\left(\Delta_1 V_t^{\lambda}(c-1)\right)$$

$$+ \left(\left(m_{N+1}\left(\Delta_1 V_t^{\lambda}(c+1-N)\right) - m_N\left(\Delta_1 V_t^{\lambda}(c+1-N)\right)\right)\right.$$

$$\left. - \left(m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right) - m_N\left(\Delta_1 V_t^{\lambda}(c-N)\right)\right)\right) + m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right)$$

$$- m_{N+1}\left(\Delta_1 V_t^{\lambda}(c+1-N)\right)$$

$$= \Delta_1 V_t^{\lambda}(c) - \Delta_1 V_t^{\lambda}(c-1) + m_2\left(\Delta_1 V_t^{\lambda}(c)\right) - m_2\left(\Delta_1 V_t^{\lambda}(c-1)\right)$$

$$+ m_N\left(\Delta_1 V_t^{\lambda}(c-N)\right) - m_N\left(\Delta_1 V_t^{\lambda}(c+1-N)\right)$$

$$\leq \Delta_1 V_t^{\lambda}(c) - \Delta_1 V_t^{\lambda}(c-1) + m_2\left(\Delta_1 V_t^{\lambda}(c)\right) - m_2\left(\Delta_1 V_t^{\lambda}(c-1)\right)$$

Last inequality follows by Lemma S1 c).

b) $N = N_{t+1,c}(l) = N_{t+1,c+1}(l) - 1$:

$$\Delta_1 V_{t+1}^{\lambda}(c+1|l) - \Delta_1 V_{t+1}^{\lambda}(c|l)$$

$$\leq \Delta_1 V_t^{\lambda}(c) - \Delta_1 V_t^{\lambda}(c-1) + m_2\left(\Delta_1 V_t^{\lambda}(c)\right) - m_2\left(\Delta_1 V_t^{\lambda}(c-1)\right)$$

$$+ \left(\left(m_{N+1}\left(\Delta_1 V_t^{\lambda}(c+1-N)\right) - m_N\left(\Delta_1 V_t^{\lambda}(c+1-N)\right)\right)\right.$$

$$\left. - \left(m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right) - m_N\left(\Delta_1 V_t^{\lambda}(c-N)\right)\right)\right) + m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right)$$

$$\leq \Delta_1 V_t^{\lambda}(c) - \Delta_1 V_t^{\lambda}(c-1) + m_2\left(\Delta_1 V_t^{\lambda}(c)\right) - m_2\left(\Delta_1 V_t^{\lambda}(c-1)\right)$$

$$+ m_{N+1}\left(\Delta_1 V_t^{\lambda}(c-N)\right)$$

$$= \Delta_1 V_t^{\lambda}(c) - \Delta_1 V_t^{\lambda}(c-1) + m_2\left(\Delta_1 V_t^{\lambda}(c)\right) - m_2\left(\Delta_1 V_t^{\lambda}(c-1)\right)$$

The last inequality follows by Lemma S1 b) (together with Remark S1 as $\Delta_1 V_t^{\lambda}(c-N) \geq l^N$

due to $N = N_{t+1,c}(l)$). The last equality follows by Remark S1 which says that $m_{N+1}\left(\Delta_1 V_t^\lambda(c-N)\right) = 0$.

c) $N = N_{t+1,c}(l) - 1 = N_{t+1,c+1}(l) - 1$

$$\begin{aligned}
&\Delta_1 V_{t+1}^\lambda(c+1|l) - \Delta_1 V_{t+1}^\lambda(c|l) \\
&\qquad \le \Delta_1 V_t^\lambda(c) - \Delta_1 V_t^\lambda(c-1) + m_2\left(\Delta_1 V_t^\lambda(c)\right) - m_2\left(\Delta_1 V_t^\lambda(c-1)\right) \\
&\qquad + \left(\left(m_{N+1}\left(\Delta_1 V_t^\lambda(c+1-N)\right) - m_N\left(\Delta_1 V_t^\lambda(c+1-N)\right)\right)\right. \\
&\qquad \left. - \left(m_{N+1}\left(\Delta_1 V_t^\lambda(c-N)\right) - m_N\left(\Delta_1 V_t^\lambda(c-N)\right)\right)\right) - m_{N+1}\left(\Delta_1 V_t^\lambda(c-N)\right) \\
&\qquad \le \Delta_1 V_t^\lambda(c) - \Delta_1 V_t^\lambda(c-1) + m_2\left(\Delta_1 V_t^\lambda(c)\right) - m_2\left(\Delta_1 V_t^\lambda(c-1)\right)
\end{aligned}$$

The last inequality follows by Lemma S1 b) and the fact that $m_2\left(\Delta_1 V_t^\lambda(c-1)\right) \ge 0$.

d) $N = N_{t+1,c}(l) - 1 = N_{t+1,c+1}(l) - 2$

$$\begin{aligned}
&\Delta_1 V_{t+1}^\lambda(c+1|l) - \Delta_1 V_{t+1}^\lambda(c|l) \\
&\qquad \le \Delta_1 V_t^\lambda(c) - \Delta_1 V_t^\lambda(c-1) + m_2\left(\Delta_1 V_t^\lambda(c)\right) - m_2\left(\Delta_1 V_t^\lambda(c-1)\right) \\
&\qquad + \left(\left(m_{N+1}\left(\Delta_1 V_t^\lambda(c+1-N)\right) - m_N\left(\Delta_1 V_t^\lambda(c+1-N)\right)\right)\right. \\
&\qquad \left. - \left(m_{N+1}\left(\Delta_1 V_t^\lambda(c-N)\right) - m_N\left(\Delta_1 V_t^\lambda(c-N)\right)\right)\right) + m_{N+2}\left(\Delta_1 V_t^\lambda(c-N)\right) \\
&\qquad - m_{N+1}\left(\Delta_1 V_t^\lambda(c-N)\right) \\
&\qquad \le \Delta_1 V_t^\lambda(c) - \Delta_1 V_t^\lambda(c-1) + m_2\left(\Delta_1 V_t^\lambda(c)\right) - m_2\left(\Delta_1 V_t^\lambda(c-1)\right)
\end{aligned}$$

The last inequality follows by Lemma S1 a) and b).

For each of the four cases, it holds that

$$\begin{aligned}
&\Delta_1 V_{t+1}^\lambda(c+1|l) - \Delta_1 V_{t+1}^\lambda(c|l) \\
&\qquad \le \Delta_1 V_t^\lambda(c) - \Delta_1 V_t^\lambda(c-1) + m_2\left(\Delta_1 V_t^\lambda(c)\right) - m_2\left(\Delta_1 V_t^\lambda(c-1)\right)
\end{aligned}$$

By replacing the optimal solution $\underline{w_2}\left(\Delta_1 V_t^\lambda(c-1)\right)$ of $m_2\left(\Delta_1 V_t^\lambda(c-1)\right)$ by the optimal solution $\underline{w_2}\left(\Delta_1 V_t^\lambda(c)\right)$ of $m_2\left(\Delta_1 V_t^\lambda(c)\right)$, we can show that

$$\begin{aligned}
&\Delta_1 V_t^\lambda(c) - \Delta_1 V_t^\lambda(c-1) + m_2\left(\Delta_1 V_t^\lambda(c)\right) - m_2\left(\Delta_1 V_t^\lambda(c-1)\right) \\
&\qquad\le \Delta_1 V_t^\lambda(c) - \Delta_1 V_t^\lambda(c-1) + \left(1 - F_\omega\left(\underline{w_2}\left(\Delta_1 V_t^\lambda(c)\right)\right)\right) \\
&\qquad\cdot \left(\Delta_1 V_t^\lambda(c-1) - \Delta_1 V_t^\lambda(c)\right) = F_\omega\left(\underline{w_2}\left(\Delta_1 V_t^\lambda(c)\right)\right) \cdot \left(\Delta_1 V_t^\lambda(c) - \Delta_1 V_t^\lambda(c-1)\right) \\
&\qquad\le 0
\end{aligned}$$

where the last inequality is a result of the induction hypothesis.

The final step is easily done:

$\Delta_1 V_{t+1}^\lambda(c+1) - \Delta_1 V_{t+1}^\lambda(c) = \int_0^1 \Delta_1 V_{t+1}^\lambda(c+1|l) - \Delta_1 V_{t+1}^\lambda(c|l) \cdot f_\lambda(l)\, dl \le 0.$ □

S.6 Proof of Proposition 7

a): We show $\Delta_1 V_t^\lambda(c) \ge \Delta_1 V_{t-1}^\lambda(c)$ for every $t \ge 1$:

$$\begin{aligned}
\Delta_1 V_t^\lambda(c|l) &= \max_{\boldsymbol{r}\in\mathcal{R}_c(l)} \left\{ \sum_{j=1}^{N_{t,c}(l)} p_j(\boldsymbol{r}|l) \cdot \left(r_j - \Delta_j V_{t-1}^\lambda(c)\right) \right\} \\
&\quad - \max_{\boldsymbol{r}\in\mathcal{R}_{c-1}(l)} \left\{ \sum_{j=1}^{N_{t,c-1}(l)} p_j(\boldsymbol{r}|l) \cdot \left(r_j - \Delta_j V_{t-1}^\lambda(c-1)\right) \right\} + \Delta_1 V_{t-1}^\lambda(c) \\
&\ge \sum_{j=1}^{N_{t,c-1}(l)} p_j(\boldsymbol{r}_t(c-1|l)|l) \cdot \left(\Delta_j V_{t-1}^\lambda(c-1) - \Delta_j V_{t-1}^\lambda(c)\right) + \Delta_1 V_{t-1}^\lambda(c) \\
&\ge \Delta_1 V_{t-1}^\lambda(c)
\end{aligned}$$

The first inequality follows by suboptimality of $\boldsymbol{r}_t(c-1|l)$ in state (t,c), the second by Proposition 6. Finally, $\Delta_1 V_t^\lambda(c) = \int_0^1 \Delta_1 V_t^\lambda(c|l) \cdot f_\lambda(l)\, dl \ge \Delta_1 V_{t-1}^\lambda(c)$.

b): Obviously, $V_t^\lambda(c)$ is increasing in t as it is never harmful to have more time to sell the remaining stock. Regarding concavity, it holds that

$V_t^{\lambda}(c|l) - V_{t-1}^{\lambda}(c|l)$

$$= \max_{\boldsymbol{r}\in\mathcal{R}_c(l)} \left\{ \sum_{j=1}^{N_{t,c}(l)} p_j(\boldsymbol{r}|l) \cdot \left(r_j - \Delta_j V_{t-1}^{\lambda}(c)\right) \right\}$$

$$- \max_{\boldsymbol{r}\in\mathcal{R}_c(l)} \left\{ \sum_{j=1}^{N_{t-1,c}(l)} p_j(\boldsymbol{r}|l) \cdot \left(r_j - \Delta_j V_{t-2}^{\lambda}(c)\right) \right\} + V_{t-1}^{\lambda}(c) - V_{t-2}^{\lambda}(c)$$

$$\leq \sum_{j=1}^{N_{t,c}(l)} p_j(\boldsymbol{r}_t(c|l)|l) \cdot \left(\Delta_j V_{t-2}^{\lambda}(c) - \Delta_j V_{t-1}^{\lambda}(c)\right) + V_{t-1}^{\lambda}(c) - V_{t-2}^{\lambda}(c)$$

$$\leq V_{t-1}^{\lambda}(c) - V_{t-2}^{\lambda}(c)$$

The first inequality follows by suboptimality of $\boldsymbol{r}_t(c|l)$ in state $(t-1, c)$, the second by Proposition 7a). Finally, $V_t^{\lambda}(c) - V_{t-1}^{\lambda}(c) = \int_0^1 \left(V_t^{\lambda}(c|l) - V_{t-1}^{\lambda}(c|l)\right) \cdot f_\lambda(l)\, dl \leq V_{t-1}^{\lambda}(c) - V_{t-2}^{\lambda}(c)$. □

S.7 Proof of Proposition 8

$V_t^{\omega,\lambda}(\cdot)$ is obviously increasing as it is never harmful to have more capacity. We show concavity by induction:

$\underline{t = 1}$:

As $V_1^{\omega,\lambda}(c|w,l) = w \cdot \sum_{i=0}^{c-1} l^i$, it holds

$$\Delta_1 V_1^{\omega,\lambda}(c+1|w,l) - \Delta_1 V_1^{\omega,\lambda}(c|w,l) = w \cdot l^c - w \cdot l^{c-1} \leq 0.$$

Subsequently, it also holds

$\Delta_1 V_1^{\omega,\lambda}(c+1) - \Delta_1 V_1^{\omega,\lambda}(c)$

$$= \int_0^1 \int_0^1 \left(\Delta_1 V_1^{\omega,\lambda}(c+1|w,l) - \Delta_1 V_1^{\omega,\lambda}(c|w,l)\right) \cdot f_\lambda(l) \cdot f_\omega(w)\, dl\, dw \leq 0$$

$\underline{t \curvearrowright t+1}$:

In the induction step, we consider four different cases regarding $N = N_{t+1,c}(w,l)$ (cf. Lemma 6):

a) $N_{t+1,c-1}(w,l) = N = N_{t+1,c+1}(w,l)$:

$\Delta_1 V_{t+1}^{\omega,\lambda}(c+1|w,l) - \Delta_1 V_{t+1}^{\omega,\lambda}(c|w,l) = \Delta_1 V_t^{\omega,\lambda}(c+1) - \Delta_1 V_t^{\omega,\lambda}(c) + \sum_{j=1}^{N} \Big(\Delta_1 V_t^{\omega,\lambda}(c +$

$1-j)-\Delta_1 V_t^{\omega,\lambda}(c+2-j)\Big)-\sum_{j=1}^{N}\Big(\Delta_1 V_t^{\omega,\lambda}(c-j)-\Delta_1 V_t^{\omega,\lambda}(c+1-j)\Big)=\Delta_1 V_t^{\omega,\lambda}(c+1-N)-\Delta_1 V_t^{\omega,\lambda}(c-N)\leq 0$

b) $N_{t+1,c-1}(w,l)+1=N=N_{t+1,c+1}(w,l)$

$\Delta_1 V_{t+1}^{\omega,\lambda}(c+1|w,l)-\Delta_1 V_{t+1}^{\omega,\lambda}(c|w,l)=\Delta_1 V_t^{\omega,\lambda}(c+1)-\Delta_1 V_t^{\omega,\lambda}(c)+\sum_{j=1}^{N}\Big(\Delta_1 V_t^{\omega,\lambda}(c+1-j)-\Delta_1 V_t^{\omega,\lambda}(c+2-j)\Big)-\sum_{j=1}^{N-1}\Big(\Delta_1 V_t^{\omega,\lambda}(c-j)-\Delta_1 V_t^{\omega,\lambda}(c+1-j)\Big)-\Big(w\cdot l^{N-1}-\Delta_1 V_t^{\omega,\lambda}(c+1-N)\Big)=\Delta_1 V_t^{\omega,\lambda}(c+1-N)-w\cdot l^{N-1}\leq 0$

c) $N_{t+1,c-1}(w,l)=N=N_{t+1,c+1}(w,l)-1$

$\Delta_1 V_{t+1}^{\omega,\lambda}(c+1|w,l)-\Delta_1 V_{t+1}^{\omega,\lambda}(c|w,l)=\Delta_1 V_t^{\omega,\lambda}(c+1)-\Delta_1 V_t^{\omega,\lambda}(c)+\Big(w\cdot l^{N}-\Delta_1 V_t^{\omega,\lambda}(c+1-N)\Big)+\sum_{j=1}^{N}\Big(\Delta_1 V_t^{\omega,\lambda}(c+1-j)-\Delta_1 V_t^{\omega,\lambda}(c+2-j)\Big)-\sum_{j=1}^{N}\Big(\Delta_1 V_t^{\omega,\lambda}(c-j)-\Delta_1 V_t^{\omega,\lambda}(c+1-j)\Big)=w\cdot l^{N}-\Delta_1 V_t^{\omega,\lambda}(c-N)<0$

d) $N_{t+1,c-1}(w,l)+1=N=N_{t+1,c+1}(w,l)-1$

$\Delta_1 V_{t+1}^{\omega,\lambda}(c+1|w,l)-\Delta_1 V_{t+1}^{\omega,\lambda}(c|w,l)=\Delta_1 V_t^{\omega,\lambda}(c+1)-\Delta_1 V_t^{\omega,\lambda}(c)+\Big(w\cdot l^{N}-\Delta_1 V_t^{\omega,\lambda}(c+1-N)\Big)+\sum_{j=1}^{N}\Big(\Delta_1 V_t^{\omega,\lambda}(c+1-j)-\Delta_1 V_t^{\omega,\lambda}(c+2-j)\Big)-\sum_{j=1}^{N-1}\Big(\Delta_1 V_t^{\omega,\lambda}(c-j)-\Delta_1 V_t^{\omega,\lambda}(c+1-j)\Big)-\Big(w\cdot l^{N-1}-\Delta_1 V_t^{\omega,\lambda}(c+1-N)\Big)=w\cdot l^{N}-w\cdot l^{N-1}\leq 0$

The inequality of a), b), c), and d) follows by induction hypothesis, definition of $N_{t+1,c}(w,l)$, definition of $N_{t+1,c}(w,l)$, and $l\in[0,1]$ respectively.

Finally, for all four cases it holds

$\Delta_1 V_{t+1}^{\omega,\lambda}(c+1)-\Delta_1 V_{t+1}^{\omega,\lambda}(c)=\int_0^1\int_0^1\Big(\Delta_1 V_{t+1}^{\omega,\lambda}(c+1|w,l)-\Delta_1 V_{t+1}^{\omega,\lambda}(c|w,l)\Big)\cdot f_\lambda(l)\cdot f_\omega(w)\,dl\,dw\leq 0.$ □

S.8 Proof of Proposition 9

a) For $t,c\geq 1$, it holds

$$\begin{aligned}\Delta_1 V_{t+1}^{\omega,\lambda}(c|w,l)-\Delta_1 V_t^{\omega,\lambda}(c)&=V_{t+1}^{\omega,\lambda}(c|w,l)-V_{t+1}^{\omega,\lambda}(c-1|w,l)-V_t^{\omega,\lambda}(c)+V_t^{\omega,\lambda}(c-1)\\&=1_{\{N_{t+1,c-1}(w,l)=N_{t+1,c}(w,l)-1\}}\cdot\Big(w\cdot l^{N_{t+1,c}(w,l)-1}-\Delta_1 V_t^{\omega,\lambda}\left(c+1-N_{t+1,c}(w,l)\right)\Big)\\&\quad+\sum_{j=1}^{N_{t+1,c-1}(w,l)}\Big(\Delta_1 V_t^{\omega,\lambda}(c-j)-\Delta_1 V_t^{\omega,\lambda}(c+1-j)\Big)\geq 0.\end{aligned}$$

The inequality follows by Lemma 5 and Proposition 8. The last step is given by $\Delta_1 V_{t+1}^{\omega,\lambda}(c) - \Delta_1 V_t^{\omega,\lambda}(c) = \int_0^1 \int_0^1 \left(\Delta_1 V_{t+1}^{\omega,\lambda}(c|w,l) - \Delta_1 V_t^{\omega,\lambda}(c)\right) \cdot f_\lambda(l) \cdot f_\omega(w)\, dw\ dl \geq 0$.

b) There is no downside in having more time to sell remaining capacity. In the contrary, there is always a positive probability for earning additional revenues by having additional selling opportunities. Thus, value function $V_t^{\omega,\lambda}(c)$ is increasing. Concavity, i.e. $V_{t+1}^{\omega,\lambda}(c) - V_t^{\omega,\lambda}(c) \leq V_t^{\omega,\lambda}(c) - V_{t-1}^{\omega,\lambda}(c)$ is given:

$$
\begin{aligned}
& V_{t+1}^{\omega,\lambda}(c|w,l) - V_t^{\omega,\lambda}(c) - V_t^{\omega,\lambda}(c|w,l) + V_{t-1}^{\omega,\lambda}(c) \\
&= \sum_{j=1}^{N_{t+1,c}(w,l)} \left(w \cdot l^{j-1} - \Delta_1 V_t^{\omega,\lambda}(c+1-j)\right) \\
&- \sum_{j=1}^{N_{t,c}(w,l)} \left(w \cdot l^{j-1} - \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)\right) \\
&= -1_{\{N_{t+1,c}(w,l)<N_{t,c}(w,l)\}} \cdot \sum_{j=N_{t+1,c}(w,l)+1}^{N_{t,c}(w,l)} \left(w \cdot l^{j-1} - \Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j)\right) \\
&+ \sum_{j=1}^{N_{t+1,c}(w,l)} \left(\Delta_1 V_{t-1}^{\omega,\lambda}(c+1-j) - \Delta_1 V_t^{\omega,\lambda}(c+1-j)\right) \leq 0
\end{aligned}
$$

with $N_{t,c}(w,l) \geq N_{t+1,c}(w,l)$ as $\Delta_1 V_t^{\omega}(c+1-j) \leq \Delta_1 V_{t+1}^{\omega}(c+1-j)$ (cf. part a) of this proposition). The inequality follows by Lemma 5 and again part a) of this proposition. Finally, $V_{t+1}^{\omega,\lambda}(c) - 2V_t^{\omega,\lambda}(c) + V_{t-1}^{\omega,\lambda}(c) = \int_0^1 \int_0^1 \left(V_{t+1}^{\omega,\lambda}(c|w,l) - V_t^{\omega,\lambda}(c) - V_t^{\omega,\lambda}(c|w,l) + V_{t-1}^{\omega,\lambda}(c)\right) \cdot f_\lambda(l) \cdot f_\omega(w)\, dl\ dw \leq 0$. □