Asymptotically Optimal Solutions for Nonlinear Dynamic Pricing in the Presence of Multiunit Demand

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Abstract

In this paper, we introduce a nonlinear dynamic pricing model in the presence of multiunit demand, enabling firms to quote separate prices for each batch size. This approach diverges from traditional models by accounting for customer heterogeneity in product attraction and batch size preference, each modeled by separate random variables in the calculation of customers' willingness-to-pay. The underlying customer choice model results in a complex formulation of purchase probabilities, necessitating considerable effort for refinements to derive a manageable expression.

After these refinements, we develop optimality conditions for the stage-wise optimization problem. As finding the optimal solution in every state remains non-trivial, we resort to formulating a fluid approximation model. Under a simplifying assumption, we can solve this approximation and subsequently verify that this assumption indeed holds for the obtained solution. The resulting static pricing policy is asymptotically optimal for our dynamic setting. Rather than directly applying this static policy, we leverage it to ensure the asymptotic optimality of three innovative heuristic methods developed within this study.

In our simulation study, we benchmark these heuristics against an upper bound and analyze patterns in the corresponding policies to gain managerial insights. Notably, our findings suggest that a piecewise linear pricing structure performs very well, offering an easy-to-communicate alternative to full nonlinear pricing.

Keywords: Revenue Management, Dynamic Pricing, Nonlinear Pricing, Multiunit Demand, Customer Choice

1 Introduction

Nonlinear pricing – such as volume discounts or special offers like "buy 3, pay 2" – were and still are a commonly applied pricing strategy in the ever-evolving field of retail. Enabled by advancements in digital technologies (e.g., e-commerce and digital price tags), businesses can now adjust prices in real-time. This allows for swift responses to varying market demands and inventory levels. However, traditional dynamic pricing models have a critical limitation: they usually assume that customers purchase only a single unit at a time. This assumption overlooks the complexities and opportunities presented by multiunit purchases, which are prevalent in sectors ranging from groceries to clothing.

Addressing this gap, our paper presents an approach that combines nonlinear and dynamic pricing to optimally quote prices for every possible batch size of a product. This framework is designed to maximize revenue in a scenario, where the selling horizon is finite and the product inventory scarce. In such a setting, the firm faces a nuanced decision-making process of customers who choose between different alternatives, comprising of different prices for varying batch sizes. Central to our approach is our novel customer choice model, specifically tailored for capturing multiunit purchasing behaviors. To achieve this, we use two random variables to model customers' willingness-to-pay.

This dual-variable approach is both novel and necessary. By considering two independent random variables, our model offers a more accurate and nuanced modeling of how differently customers value multiple units of a product. This innovative perspective is essential for businesses to dynamically adjust batch prices in real-time, ensuring that pricing strategies are not only responsive but also anticipative with regards to customer behavior.

We contribute on the sparse literature on multiunit dynamic pricing by addressing the complex optimization challenge of dynamically quoting batch prices. Due to the complex customer choice model, deriving a closed-form solution to our optimization problem is intractable and beyond the scope of this paper. Nonetheless, our analysis sheds light on the structural properties and optimality conditions inherent in the multiunit dynamic pricing problem, laying a foundation for heuristic solution methods that approximate optimal outcomes. An important achievement of our research is ensuring asymptotic optimality, a desired attribute for heuristic methodologies, which we derive from the analysis of a fluid approximation.

However, this fluid approximation still faces the difficulties posed by the complex customer choice model. To navigate these challenges, we propose a modified fluid model that, under certain conditions, provides the original approximation's solution while being significantly easier to solve. We introduce three innovative algorithms that, while heuristic, approach optimality asymptotically. Two of these algorithms build on the foundational work of Schur (2024), based on observed customer information. The third algorithm proposes a novel decomposition approach, simplifying the customer choice model and offering an alternative, numerically solvable optimization problem that serves as a proxy for our original model.

This paper is organized as follows: In Section 2, we briefly review relevant literature. Section 3 details our new customer choice model, emphasizing the critical role of our dual-variable approach, and presents the optimization model. Section 4 commences with reducing complexity of the probability function, enabling the determination of optimality conditions on an optimal solution. Subsequently, a fluid model as well as a modified fluid model are developed, both to motivate the asymptotic optimality of our heuristic algorithms, presented in Section 5. In Section 6, we present a numerical study that underlines the efficiency of our heuristics and provides managerial insights, establishing the practical benefits of adopting an intricate, nonlinear dynamic pricing strategy.

2 Literature review

In this study, we bridge the domains of dynamic pricing and nonlinear pricing, integrating the strengths of both. Initially, we start with a concise review of works within these two distinct yet interconnected domains. This foundational overview sets the stage for a deeper exploration into the specialized segments of multiunit and multiproduct dynamic pricing.

Multiunit dynamic pricing, which also covers nonlinear pricing, represents a relatively new field of research with sparse literature. Our research contributes to this niche, acknowledging the critical role of nonlinear pricing in catering to multiunit demands. Following this, we shift our focus to multiproduct dynamic pricing, a domain aligned with our research due to its focus on customer choice models. Here, customers are presented with multiple options, mirroring the decision-making process central to our study.

Nonlinear pricing is a widespread strategy across various sectors, including telecommunications, transportation, energy, supply chains, and retail. Consequently, there is a rich and diverse literature on the subject. R. Wilson (1993) provides a comprehensive overview of the application fields, economic principles, and marketing insights related to nonlinear pricing. While much of the existing literature focuses on static pricing models, a small subset of researchers has turned their attention to dynamic environments, which align more closely with the thematic focus of our study (e.g., Dhebar & Oren, 1986, and Braden & Oren, 1994).

The conceptual foundations of dynamic pricing trace back to seminal studies on intertemporal price discrimination conducted 30 to 40 years ago, with notable contributions by Stokey (1979), Landsberger and Meilijson (1985), and C. A. Wilson (1988). A significant milestone was achieved by Gallego and van Ryzin (1994), who were the first to explore optimal dynamic pricing for a single product under stochastic demand over a finite selling horizon. This pioneering work led to a vast amount of follow-up research, which was reviewed and summarized by many authors, including Bitran and Caldentey (2003), Chiang et al. (2007), and, with a special focus, Gönsch et al. (2013) and den Boer (2015), as well as in textbooks by Talluri and van Ryzin (2004) (Chapter 5) and Phillips (2005) (Chapter 10).

While dynamic pricing has been extensively explored, the specific area of multiunit dynamic pricing is still emerging. Elmaghraby et al. (2008) delve into markdown pricing mechanisms within a multiunit demand framework, assuming complete information about customers and their willingness to pay. Levin et al. (2014) expanded the discussion to a dynamic pricing model characterized by stochastic customer demand for batches. In their framework, customers request a specific batch size and the seller's subsequent pricing response determines whether a purchase is made. Thereby, the probability of purchase is influenced by the price set for the batch. However, unlike their approach, our research introduces a more flexible decision-making process where customers can review all available prices prior to determining their optimal purchase quantity. This advancement not only empowers customers with greater choice but also provides firms with a mechanism to strategically steer customers' buying decisions and purchase volumes.

Gallego et al. (2020) investigate three dynamic pricing strategies: nonlinear, linear, and block pricing. In their proposed choice model, customers seek to maximize their utility and are characterized by a single random variable. The authors develop optimality conditions and show structural properties. However, our research diverges fundamentally by modeling customer behavior with two independent variables, thereby offering a more granular representation of customer decision-making processes. This dual-variable approach represents a significant departure from the conventional single-variable models. Schur (2024) explores a scenario remarkably similar to ours but diverges in assuming firms' access to some or all private information about arriving customers. This assumption paves the way for personalized pricing strategies and the evaluation of the strategic value of customer information. Contrary to this, our research operates under the premise of limited information, enabling the universal applicability of our pricing model without relying on the availability of detailed customer insights.

Multiunit dynamic pricing can be compared to the better-explored field of multiproduct dynamic pricing by defining batches of a single product as several different "products". Several articles (see, e.g., Zhang & Cooper, 2009, Dong et al., 2009, and Akçay et al., 2010, or, for a review, Chen & Chen, 2015) have investigated dynamic pricing of substitutes. In these publications, a customer can choose between several products and each of these products has its (own) stock. Although multiunit dynamic pricing is related to multiproduct dynamic pricing of substitutes, the inventory structure often differs. Several analytical difficulties arise in a setting where "products" consume a different amount of a single resource compared to a setting where each product has its independent resources. One exception to the product-specific inventory setting is Maglaras and Meissner (2006). They consider a slightly different multiproduct model where each product consumes one unit of a single resource. They show that, in their setting, dynamic pricing and capacity control can be reduced to a common formulation in which the firm controls the consumption rate of every product regarding resource capacity. Moreover, they prove that the solution of a fluid model is asymptotically optimal. This proof is generalized to a setting where products can consume more than just one unit of a single resource.

Our comprehensive literature review highlights a significant gap in research concerning nonlinear dynamic pricing, despite the contributions of Gallego et al. (2020) and Schur (2024). These investigations, while invaluable, present methodologies distinct from our own, underscoring the novelty of our approach. Gallego et al. (2020) adopt a choice model which simplifies customer behavior to a singular random variable. In contrast, our method advances this by incorporating dual random variables, a refinement that allows for a richer, more nuanced depiction of customer decision-making. This dual-parameter approach not only provides a deeper insight into individualized customer behavior but also introduces a layer of complexity to the optimization by integrating additional dimensions of uncertainty. Conversely, Schur (2024) explores a customer choice model akin to ours, with a crucial distinction in their assumption about data accessibility. Unlike their premise of (partial) visibility into customer's private data, our study assumes no observability, ensuring the universal applicability of our pricing model without relying on the availability of detailed customer insights.

Moreover, our review indicates that other contributions to the field tend to deviate significantly in basic assumptions, exploring different scenarios altogether. Notably, many of these studies overlook the dynamics of customers with flexible multiunit demand, thereby missing the opportunity to assess how nonlinear pricing strategies can effectively shape and capitalize on stochastic purchasing behaviors.

3 Problem definition

In Section 3.1, we introduce the general setting and notation. In Section 3.2, we present the customer choice model, deriving a functional formulation of the selling probabilities. Lastly, in Section 3.3, we present the optimization model.

3.1 General setting and notation

We adapt the standard setting of dynamic pricing to cope with multiunit purchases. To do so, we consider the following framework: A firm sells a single product over a finite selling horizon. The selling horizon is divided into T periods and indexed backward in time, i.e., periods T and 0 mark the beginning and the end of the horizon, respectively. The initial stock of C units of the product is nonreplenishable and any capacity remaining after the selling horizon is worthless. We assume that exactly one customer arrives in each period $t \in \{T, \dots, 1\}$. At this moment, the firm knows the remaining capacity $c \in \{1, \dots, C\}$ and sets a price vector $\boldsymbol{r} = (r_1, r_2, \dots, r_c)^T$ with r_j marking the price the customer has to pay for j units of the product, i.e. for a batch with batch size $j \leq c$. Depending on the prices the firm quotes, this customer may purchase zero, one or more (up to c) units of the product. Thereby, $p_j(\boldsymbol{r})$ denotes the probability that this customer chooses to buy j units of the product.

3.2 Customer choice model

In our model, customers are presented with several purchasing options, ranging from various batch sizes to opting not to purchase at all (a batch size of zero). Each customer internally evaluates these options based on a personal, albeit unknown to the firm, willingness-to-pay, which can be quantified monetarily. The decision-making process hinges on the utility derived from each option – the difference between the customer's willingness-to-pay and the batch price. The optimal choice for the customer is the one that maximizes this utility. This conceptual framework is widely recognized in the fields of economics and pricing literature for its effectiveness in capturing the heterogeneity of customer preferences (through individual willingness-to-pay) and the firm's influence on customer choices (through pricing). Grounded in economic theory, this model assumes that individuals' decisions are motivated by maximizing perceived value against cost, a particularly apt approach when customers are faced with multiple options beyond a mere binary buy-or-not decision. This methodology is often applied in literature, including in the work of Braden and Oren (1994), who examined nonlinear (static) pricing, and Akçay et al. (2010), who explored multiproduct dynamic pricing.

Each customer's willingness-to-pay, denoted as X_j for a batch of size j, is considered private information and thus remains unknown to the firm, rendering X_j a random variable. Furthermore, a common assumption in literature, found e.g. in Baucells and Sarin (2007), Goldman et al. (1984), Iyengar and Jedidi (2012), and Gallego et al. (2020), is that the marginal willingness-to-pay, represented by $X_{j+1} - X_j$, is non-negative and diminishes with each additional unit. This structure includes the notion that while each additional unit is valued, it is less so than its predecessor. We incorporate this critical aspect into our model to realistically simulate customer behavior in scenarios involving batch purchases.

We employ a model inspired by Iyengar and Jedidi (2012), further refined by Schur (2024), to capture this phenomenon. Iyengar and Jedidi (2012) propose a willingness-to-pay function based on known parameters, incorporating an error term to account for uncertainty in customer behavior. Schur (2024) builds on this by treating the parameters themselves as random variables, representing private information. Thus, randomness is introduced directly into the model parameters rather than through an error term. Our approach adopts this latter perspective, defining the willingness-to-pay X_j for a batch size of j by:

$$X_j = \omega \cdot \sum_{k=0}^{j-1} (\lambda)^k \qquad \text{for } j = 1, \dots, c, \tag{1}$$

with independent, continuous, and time-homogeneous random variables ω and λ. In our model, we characterize the density functions of the random variables ω and λ as f_ω and f_λ, respectively. Correspondingly, their cumulative distribution functions are denoted by F_ω and F_λ, with both density functions supported on the interval $[0, 1]$.

By constraining λ within the range $[0, 1]$, we ensure that the marginal willingness-to-pay, defined as $X_{j+1} - X_j = \omega \cdot \lambda^j$, remains non-negative and is decreasing with respect to the quantity j, assuming $\omega \geq 0$. This specification aligns with the earlier-stated common assumption about customer preferences, effectively capturing the diminishing value of additional units. The limitation of ω to $[0, 1]$ serves primarily as a scaling measure, normalizing the marginal willingness-to-pay.

Regarding the interpretation of the random variables ω and λ: ω equals the willingness-to-pay for the first unit, evident from $X_1 = \omega$ (since $X_1 = \omega \cdot \sum_{k=0}^{0}(\lambda)^k = \omega$), and linearly affects $X_j = \omega \cdot \left(\sum_{k=0}^{j-1}(\lambda)^k\right)$ for $j \geq 2$. Thus, ω can be perceived as the product's attractiveness to the customer, or more formally, as the base willingness-to-pay. Conversely, λ does not impact $X_1 = \omega$ but plays a crucial role in determining the rate at which the marginal willingness-to-pay diminishes with increasing j. This is illustrated by the relationship $X_{j+1} - X_j = \omega \cdot \lambda^j = \lambda \cdot (\omega \cdot \lambda^{j-1}) = \lambda \cdot (X_j - X_{j-1})$, allowing us to interpret λ as a measure of the customer's inclination to stockpile or consume. Essentially, λ acts as an indicator of consumption patterns, reflecting how the value customers place on additional units decreases with the batch size.

Our approach to modeling customer choice is anchored in the widely accepted random utility model. Within this framework, we define a customer's utility for purchasing j units at given prices as the difference between their random willingness-to-pay and the corresponding price. Specifically, for a price vector $\boldsymbol{r}$, the utility of purchasing j units, denoted $u_j(\boldsymbol{r})$, is calculated as:

$$u_j(\boldsymbol{r}) = X_j - r_j \qquad \text{for } j = 1, \dots, c. \tag{2}$$

Under the assumption of rational behavior, customers aim to maximize their utility. Therefore, a customer opts to purchase j units if, and only if, $u_j(\boldsymbol{r})$ represents the maximum utility achievable across all possible purchase quantities, including the option not to purchase ($u_0(\boldsymbol{r}) = 0$). This modeling assumption, in conjunction with the density functions for ω and λ, allows us to compute the probability of a customer purchasing j units. This computation involves determining the probability that a specific realization of the random variables ω and λ, which represent a unique customer profile, results in the maximum utility for the purchase of batch size j.

To visualize this concept, one can imagine mapping out decision regions within the $[0, 1]^2$ space, identified by combinations of w and l (realizations of ω and λ, respectively), where the utility of purchasing j units surpasses that of any other purchase quantity within the set $\{0, 1, \dots, c\}$. These regions are effectively determined by where $u_j(\boldsymbol{r})$ attains its maximum value across all considered purchasing options ($u_j(\boldsymbol{r}) = \max_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}$). Subsequently, the probability associated with ω and λ falling within a particular region reflects the likelihood of a customer, characterized by that specific (w, l) bundle, opting to purchase j units. The following figure illustrates these regions, with each one labeled according to the batch size j that maximizes utility for customers whose preferences are represented by the (w, l) bundles within that region.

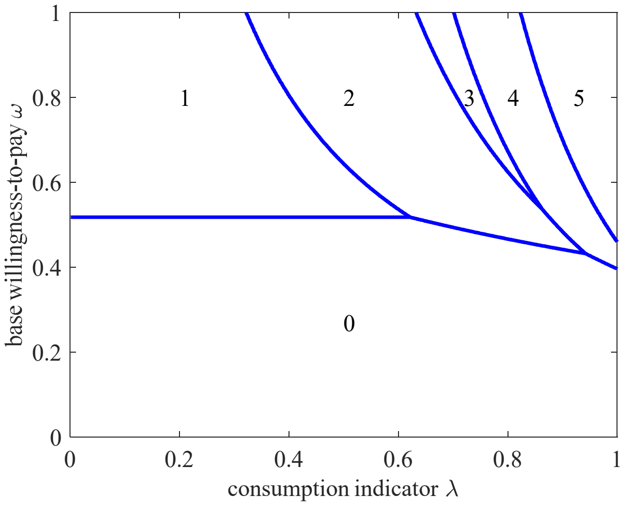

Figure 1: An example of the location of decision regions for $c = 5$

Consequently, the choice probability for a customer to purchase j units can be calculated by

$$p_j(\boldsymbol{r}) = \int_0^1 \int_0^1 1_{\left\{u_j(\boldsymbol{r}) = \max\limits_{j=0,\dots,c} \{u_j(\boldsymbol{r})\}\right\}}(w, l)\, f_\omega(w)\, f_\lambda(l)\, dw\, dl \qquad \text{for } j = 1, \dots, c, \tag{3}$$

where the indicator function $1_{\left\{u_j(\boldsymbol{r}) = \max\limits_{j=0,\dots,c} \{u_j(\boldsymbol{r})\}\right\}}(w, l)$ equals 1 if the utility of purchasing j units, $u_j(\boldsymbol{r})$, is the maximum among all considered batch sizes. Specifically, this condition is satisfied if the utility of purchasing j units, $w \cdot \sum_{k=0}^{j-1} l^k - r_j$, is greater than or equal to the utility of purchasing any other quantity i, $w \cdot \sum_{k=0}^{i-1} l^k - r_i$, for $i = 1, \dots, c$, and also nonnegative. If these conditions are not met, the indicator function is 0.

Alternatively, these selection criteria can be expressed through the relationship between price differences and the cumulative discounting effect of purchasing additional units. A customer with attributes $w, l \in [0,1]$ will purchase j units iff the following condition holds:

$$\max_{0 \le k \le j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i} \right\} \le w \le \min_{j+1 \le k \le c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}, 1 \right\}.$$

This expression separates both attributes w and l and simplifies calculating the choice probabilities. Furthermore, the fractions within this expression define the decision regions as they mark the boundaries of these regions (represented as blue lines in Figure 1).

To keep notation as short as possible, we further establish the following conventions: $r_0 = 0$, $\frac{r_1 - r_0}{\sum_{i=0}^{0} l^i} = r_1$ for $l = 0$, $\max\limits_{0 \le k \le j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} 0^i} \right\} = \infty$ for $j \neq 1$, and $\min\limits_{j+1 \le k \le c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}, 1 \right\} = 1$ for $j = c$.

By utilizing the alternative formulation, we can simplify the choice probability for selling j units to

$$p_j(\boldsymbol{r}) = \int_0^1 \left(F_\omega \left(\min_{j+1 \le k \le c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}, 1 \right\} \right) - F_\omega \left(\max_{0 \le k \le j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i} \right\} \right) \right)^+ f_\lambda(l)\, dl \quad \text{for } j = 1, \dots, c, \tag{4}$$

with $(\cdot)^+ = \max\{\cdot, 0\}$. While this formulation features only one integral, it remains analytically challenging. Without further knowledge, particularly about which fractions yield the minimum and maximum for any given l, even calculating the purchase probability for a given price vector r becomes a tedious task, rendering finding the optimal solution impossible. Consequently, we will put considerable effort in refining the formulation and in learning inherent structural properties in Section 4.

3.3 Dynamic programming formulation

In traditional singleunit dynamic pricing, a firm maximizes total expected revenue over the entire selling horizon:

$$V_T^{SU}(C) = \max_{r_{1t} \ge 0\ \forall t \in \{1,\dots,T\}} \left\{ \mathbb{E}\left[\sum_{t=1}^T r_{1t} \cdot 1_{\{w \ge r_{1t}\}}(w) \right] : \sum_{t=1}^T 1_{\{w \ge r_{1t}\}}(w) \le C \text{ a.s.} \right\}. \tag{5}$$

This optimization problem can be reformulated as the following dynamic program (see, e.g., Talluri & van Ryzin, 2004):

$$V_t^{SU}(c) = \max_{r_1 \ge 0} \left\{ p_1(r_1) \cdot \left(r_1 + V_{t-1}^{SU}(c-1) \right) + \left(1 - p_1(r_1)\right) \cdot V_{t-1}^{SU}(c) \right\}, \tag{6}$$

with boundary conditions $V_0^{SU}(c) = 0$ for $c \ge 0$ and $V_t^{SU}(0) = 0$ for $t \ge 0$. Here, $V_t^{SU}(c)$ denotes the optimal expected revenue-to-go from period t onwards. The expectation captures two possible events: A sale of one unit occurs with probability $p_1(r_1)$ and the firm immediately obtains a revenue of r_1 and additionally expects a revenue of $V_{t-1}^{SU}(c-1)$ with a reduced stock of $c-1$ units from the next period onwards. No sale occurs with probability $1 - p_1(r_1)$. In this case, the firm expects a revenue of $V_{t-1}^{SU}(c)$ from stock c.

Building on the formulations (5) and (6), we can develop the corresponding multiunit dynamic pricing formulation. As the remaining capacity c (and, thus, the maximum number of purchasable units) varies over time, we define a state-dependent action space $\mathcal{R}_c = \{\boldsymbol{r} \in \mathbb{R}^c : r_j \ge 0, j = 1, \dots, c\}$ with $\mathcal{R}_0 = \emptyset$. The overall target is still the maximization of the total expected revenue:

$$V_T(C) = \max_{r_t \in \mathcal{R}_C\ \forall t \in \{1,\dots,T\}} \left\{ \mathbb{E}\left[\sum_{t=1}^T \sum_{j=1}^C r_{jt} \cdot 1_{\left\{u_{jt}(r_t) = \max\limits_{j=0,\dots,c}\{u_{jt}(r_t)\}\right\}}(w,l) \right] : \sum_{t=1}^T \sum_{j=1}^C j \cdot 1_{\left\{u_{jt}(r_t) = \max\limits_{j=0,\dots,c}\{u_{jt}(r_t)\}\right\}}(w,l) \le C \text{ a.s.} \right\} \tag{7}$$

and can be achieved by using the following dynamic program:

$$V_t(c) = \max_{\boldsymbol{r} \in \mathcal{R}_c} \left\{ \sum_{j=1}^c p_j(\boldsymbol{r}) \cdot \left(r_j + V_{t-1}(c-j) \right) + \left(1 - \sum_{j=1}^c p_j(\boldsymbol{r})\right) \cdot V_{t-1}(c) \right\}, \tag{8}$$

with boundary conditions $V_0(c) = 0$ for $c \geq 0$ and $V_t(0) = 0$ for $t \geq 0$. Again, $V_t(c)$ denotes the optimal expected revenue-to-go from period t onwards (before the arrival of a customer in t).

With the new formulation, we consider that customers might be willing to buy more than just one unit. Moreover, we allow the firm to set batch prices to take full advantage of a nonlinear pricing scheme. The expectation now captures $c + 1$ possible events: With probability $p_j(\boldsymbol{r})$, the firm sells j units and immediately earns r_j. Additionally, it can expect future revenues amounting to $V_{t-1}(c - j)$ with a reduced stock of $c - j$ units from the next period onwards. With probability $1 - \sum_{j=1}^{c} p_j(\boldsymbol{r})$, the firm sells nothing and only faces expected revenue $V_{t-1}(c)$ from stock c and period $t - 1$ onwards. We denote the optimal batch prices selected in a state (t, c) by $\boldsymbol{r}_t(c) \in \mathcal{R}_c$.

An alternative formulation of (8) relies on opportunity costs $\Delta_j V_t(c)$ for selling j units, i.e.

$$\Delta_j V_t(c) = V_t(c) - V_t(c - j) \qquad \text{for } j = 1, \dots, c, \tag{9}$$

and can be written as

$$V_t(c) = \max_{\boldsymbol{r} \in \mathcal{R}_c} \left\{ \sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c) \right) \right\} + V_{t-1}(c). \tag{10}$$

This formulation presents several benefits compared to Equation (8). The most immediate benefit is the clear indication that optimal prices must exceed opportunity costs. Not meeting this criterion could mean that sales do not contribute to an increase in expected revenue, or worse, could result in a decrease in overall expected revenue. Furthermore, this formulation highlights the critical role of opportunity costs. As the only component that varies with state, opportunity costs are fundamentally responsible for the dynamic adjustments in optimal pricing strategies over time.

Summarizing, the optimization breaks down to maximizing the expected additional gain realized by selling something between 1 and c units in period t instead of retaining the capacity for later sales.

4 Optimality condition, fluid approximation and asymptotic optimal solution

In Sections 4.1 and 4.2, our objective is to streamline the formulation of the probability function in two steps. Initially, we introduce appropriate notation and show inherent structural properties, thereby formulating a more concise function. Subsequently, we eliminate prices from the action space that do not contribute to solving the optimization problem. This allows us to gain additional insights in the probability function, leading to a further refinement.

This refinement enables us to establish optimality conditions for solving the state-wise optimization problem. However, these conditions, while essential, prove insufficient for effectively computing a solution across the entire sales period. In this context, we introduce a fluid approximation and discuss the asymptotic optimality of its solution, a desirable feature we want to preserve for our heuristic methods outlined in Section 5. Recognizing that solving the fluid approximation remains complex, we propose a modified model that is significantly simpler to solve. This alternative model yields the same optimal solution under certain conditions that are straightforward to verify.

4.1 Refining the probability function

Despite simplifying the model by eliminating one of the integrals at the end of Section 3.2, the resulting probability function remains challenging to manage. Specifically, incorporating the minimum and maximum functions throughout the analysis introduces significant complexity. To address this, we propose dividing the integration region into distinct segments. This approach allows us to streamline the calculation process by substituting $\min_{j+1\leq k\leq c}\left\{\frac{r_k-r_j}{\sum_{i=j}^{k-1}l^i},1\right\}$ and $\max_{0\leq k\leq j-1}\left\{\frac{r_j-r_k}{\sum_{i=k}^{j-1}l^i}\right\}$ with more manageable expressions for each segment.

We start with introducing the following sets, designed to sort l in such a way that on each set the minimum or maximum operator can be substituted:

$$\Lambda_j(\boldsymbol{r})=\left\{l\in[0,1]\mid \max_{0\leq k\leq j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1}l^m}\right\}\leq 1=\min_{j+1\leq k\leq c}\left\{\frac{r_k-r_j}{\sum_{m=j}^{k-1}l^m},1\right\}\right\},$$

$$\Lambda_{ij}^{min}(\boldsymbol{r})=\left\{l\in[0,1]\mid \max_{0\leq k\leq j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1}l^m}\right\}\leq \frac{r_i-r_j}{\sum_{m=j}^{i-1}l^m}=\min_{j+1\leq k\leq c}\left\{\frac{r_k-r_j}{\sum_{m=j}^{k-1}l^m},1\right\}\right\},\ i>j,\text{ and}$$

$$\Lambda_{ji}^{max}(\boldsymbol{r})=\left\{l\in[0,1]\mid \max_{0\leq k\leq j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1}l^m}\right\}=\frac{r_j-r_i}{\sum_{m=i}^{j-1}l^m}\leq \min_{j+1\leq k\leq c}\left\{\frac{r_k-r_j}{\sum_{m=j}^{k-1}l^m},1\right\}\right\},\ i<j.$$

These sets are either empty or an interval. This follows immediately from Lemma 1 and is illustrated in Figure 2. In this figure, we can identify specific intervals for the sets under consideration for $j=1$: $\Lambda_{10}^{max}(\boldsymbol{r})=[0,1]$, $\Lambda_1(\boldsymbol{r})=[0,0.4]$, $\Lambda_{21}^{min}(\boldsymbol{r})=[0.4,0.55]$, $\Lambda_{31}^{min}(\boldsymbol{r})=[0.55,0.826]$, $\Lambda_{41}^{min}(\boldsymbol{r})=[0.826,1]$, and $\Lambda_{51}^{min}(\boldsymbol{r})=[1,1]$, with their boundaries marked by black lines in the figure. Many of these intervals (particularly, $\Lambda_{i1}^{min}(\boldsymbol{r})$ with $i>1$) are defined by the intersection points of the blue and green lines, signifying where the line that constitutes the minimum shifts. To calculate the probability of selling a single unit batch ($j=1$) in this scenario, it's necessary to evaluate the probability of realizations (w,l) falling within the region above the red line and beneath all green and blue lines. By identifying these specific sets, we can precisely ascertain which among the green and blue lines determines the minimum solution for any given realization l, thus identifying the corresponding realization w that result in the sale of one unit.

Lemma 1 *For every $i\neq k$ with $i,k>j$ and $r_i-r_j\neq 0\neq r_k-r_j$, there is at most one $l\in(0,1)$ where $\frac{r_i-r_j}{\sum_{m=j}^{i-1}l^m}=\frac{r_k-r_j}{\sum_{m=j}^{k-1}l^m}$. For every $i\neq k$ with $i,k<j$ and $r_i-r_j\neq 0\neq r_k-r_j$, there is at most one $l\in[0,1]$ where $\frac{r_j-r_k}{\sum_{m=k}^{j-1}l^m}=\frac{r_j-r_i}{\sum_{m=i}^{j-1}l^m}$.*

Proof: See Supplement S.1.

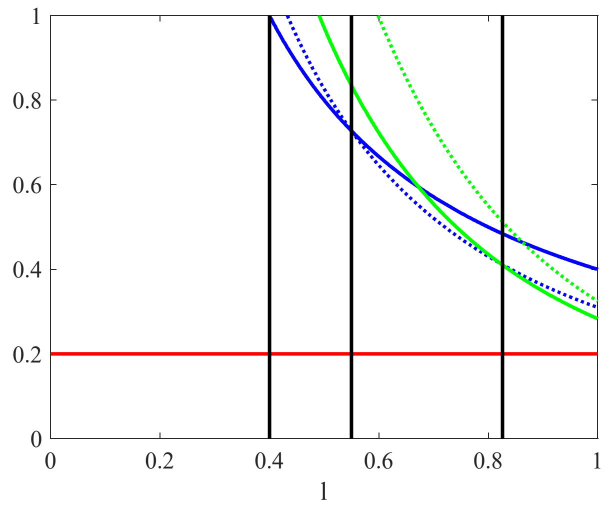

Figure 2: An example of intervals $\boldsymbol{\Lambda_j(r)}$ and $\boldsymbol{\Lambda_{ij}^{min}(r)}$ with $\boldsymbol{j = 1}$ and $\boldsymbol{c = 5}$

Lemma 1, in conjunction with the observation that the equation $\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} = 1$ is satisfied by at most one $l \in [0,1]$, implies that each curve can only serve as the minimum or maximum exclusively within a specific segment. At some point, it will intersect with another curve and will consistently remain either above or below the intersecting curve. Consequently, for the various sets of interest – namely $\Lambda_j(\boldsymbol{r})$ for instances where $1 = \min_{j+1 \le k \le c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \right\}$, $\Lambda_{ij}^{min}(\boldsymbol{r})$ for cases where $i > j$, and $\Lambda_{ji}^{max}(\boldsymbol{r})$ for cases where $j > i$ – we define their corresponding intervals as $\left[\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r})\right]$, $\left[\underline{l}_{ij}^{min}(\boldsymbol{r}), \bar{l}_{ij}^{min}(\boldsymbol{r})\right]$, and $\left[\underline{l}_{ji}^{max}(\boldsymbol{r}), \bar{l}_{ji}^{max}(\boldsymbol{r})\right]$, respectively. It's important to note that some of these intervals may be empty, and thus, boundary values have to be selected accordingly.

Remark 1 *By definition, there is* $\bar{l}_j^{end}(\boldsymbol{r}) \in [0,1]$ *such that* $\bar{l}_j^{end}(\boldsymbol{r}) = \max_{i>j} \left\{ \bar{l}_{ij}^{min}(\boldsymbol{r}) \right\} = \max_{i<j} \left\{ \bar{l}_{ji}^{max}(\boldsymbol{r}) \right\}$. *Moreover, it holds that:*

1. $\bigcup_{i>j} \left[\underline{l}_{ij}^{min}(\boldsymbol{r}), \bar{l}_{ij}^{min}(\boldsymbol{r})\right] = \left[\bar{l}_j(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$
2. $\bigcup_{i<j} \left[\underline{l}_{ji}^{max}(\boldsymbol{r}), \bar{l}_{ji}^{max}(\boldsymbol{r})\right] = \left[\underline{l}_j(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$

Currently, our model requires extensive notation to accurately articulate the probability function. Nevertheless, the introduction of the subsequent lemma will significantly streamline the notation required, thereby enhancing the brevity and clarity of our presentation.

Lemma 2 *For every $i > j$ it holds that* $\left[\underline{l}_{ij}^{min}(\boldsymbol{r}), \bar{l}_{ij}^{min}(\boldsymbol{r})\right] = \left[\underline{l}_{ij}^{max}(\boldsymbol{r}), \bar{l}_{ij}^{max}(\boldsymbol{r})\right]$.

Proof: See Supplement S.2.

This lemma not only streamlines our notation but also carries another implication: for $i > j$, the interval $\left[\underline{l}_{ij}^{min}(\boldsymbol{r}), \bar{l}_{ij}^{min}(\boldsymbol{r})\right]$ defines the region for realization l where the line $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}$ marks the upper bound for all realizations w that specify, in combination with l, all customers that opt to purchase j units. As a consequence of Lemma 2, within the same interval, the same line also establishes the lower bound for all realizations w that define, in combination with l, all customers that prefer purchasing i units. Thus, if this interval is not empty, then this line marks the boundary between opting for j units versus i units.

We can now shorten our notation to $\underline{l}_{ij}(\boldsymbol{r}) = \underline{l}_{ij}^{min}(\boldsymbol{r}) = \underline{l}_{ij}^{max}(\boldsymbol{r})$ and $\bar{l}_{ij}(\boldsymbol{r}) = \bar{l}_{ij}^{min}(\boldsymbol{r}) = \bar{l}_{ij}^{max}(\boldsymbol{r})$. The probability function can be written as:

$$p_j(\boldsymbol{r}) = \int_{\underline{l}_j(r)}^{\bar{l}_j(r)} f_\lambda(l)\, dl + \sum_{i=j+1}^{c} \int_{\underline{l}_{ij}(r)}^{\bar{l}_{ij}(r)} F_\omega\left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}\right) f_\lambda(l)\, dl - \sum_{i=0}^{j-1} \int_{\underline{l}_{ji}(r)}^{\bar{l}_{ji}(r)} F_\omega\left(\frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m}\right) f_\lambda(l)\, dl \qquad \text{for } j = 1, \dots, c. \tag{11}$$

For now, our refinement of the probability function reaches its limit without further specifics on the distribution function F_ω or the structure of possible price vector $\boldsymbol{r}$. Given our reluctance to (prematurely) narrowing our focus to a particular distribution function for ω, we shift our attention towards simplifying the probability function by examining feasible price vectors. In this endeavor, our goal is to eliminate any price vector that does not contribute to our optimization objective, thereby gaining additional insights that refine the formulation of the probability function.

4.2 Action space reduction

Upon closer examination of our choice model, it becomes evident that we only need to consider a specific subset of prices to maximize expected revenue. Consequently, we aim to refine the definition of the action space $\mathcal{R}_c$ by excluding prices that have no impact on our optimization problem.

The argumentation for deeming certain prices as irrelevant is as follows: Maintaining multiple prices r_j that effectively nullify demand for j units (i.e., $p_j(\boldsymbol{r}) = 0$) is unnecessary. It suffices to have a single r_j (depending on $r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_c$) to preserve the option of pricing out j units. Our argumentation on determining irrelevant prices follows four steps.

1. **Exclusion of higher prices for smaller batches:** We exclude any price r_j with $r_j > r_{j+1}$ because customers almost surely have a higher willingness-to-pay for $j + 1$ units than for j units ($X_{j+1} - X_j = \omega \cdot \lambda^j \geq 0$). Consequently, customers would not pay a higher price for j units than for $j + 1$ units. Hence, for $r_j > r_{j+1}$, it follows that $p_j(\boldsymbol{r}) = 0$. However, the same effect can be achieved by setting $r_j = r_{j+1}$, making prices $\boldsymbol{r}$ with $r_j > r_{j+1}$ irrelevant.
2. **Exclusion of prices exceeding batch size threshold:** Any price r_j exceeding j is irrelevant. Given that $\omega \sum_{k=0}^{j-1} \lambda^k \leq j \leq r_j$ implies that $p_j(\boldsymbol{r}) = 0$, we can drop $r_j > j$ and still have the possibility to achieve $p_j(\boldsymbol{r}) = 0$ by setting $r_j = j$.

3. **Exclusion of prices with excessive margins:** We exclude any price r_j for which $r_j - r_{j-1} > 1$. The rationale behind dismissing these prices is grounded in the following inequality: $\max_{0\le k\le j-1}\left\{\frac{r_j-r_k}{\sum_{i=k}^{j-1}\lambda^i}\right\} \ge \frac{r_j-r_{j-1}}{\lambda^{j-1}} \ge 1 \ge \min_{j+1\le k\le c}\left\{\frac{r_k-r_j}{\sum_{i=j}^{k-1}\lambda^i}, 1\right\}$, which emerges from the condition $r_j - r_{j-1} > 1$. Consequently this leads to $p_j(\boldsymbol{r}) = \left(\min_{j+1\le k\le c}\left\{\frac{r_k-r_j}{\sum_{i=j}^{k-1}\lambda^i}, 1\right\} - \max_{0\le k\le j-1}\left\{\frac{r_j-r_k}{\sum_{i=k}^{j-1}\lambda^i}\right\}\right)^+ = 0$. Therefore, maintaining r_j such that $r_j - r_{j-1} = 1$ in our action space suffices to nullify demand for j units if desired.
4. **Exclusion of prices with excessive comparative margins:** We omit prices r_j that satisfy the condition $(r_j - r_{j-1})^{\frac{1}{j-1}} > (r_{j+1} - r_j)^{\frac{1}{j}}$. This inequality implies that a customer with a positive marginal utility for purchasing the jth unit has almost certainly also a positive marginal utility for purchasing the $j+1$th unit. This is verified by the proof of the Lemma 3.

By systematically excluding these prices, we refine our pricing strategy to focus only on those prices that impact the solution of our optimization problem.

Lemma 3 *Relevant prices* r_j *are given by* $\mathcal{R}_c = \Big\{\boldsymbol{r} \in \mathbb{R}^c\colon 0 \le r_1 \le \cdots \le r_c \le c,\ r_j \le j\ \forall j, r_j - r_{j-1} \le 1\ for\ j \ge 2, and\ (r_j - r_{j-1})^{\frac{1}{j-1}} \le (r_{j+1} - r_j)^{\frac{1}{j}}\ for\ 2 \le j \le c-1\Big\}$.

Proof: See Supplement S.3.

In Section 3.2, we have seen that $\underline{l}_k$ and $\bar{l}_k$ play a crucial role in calculating selling probabilities. With the action space reduction, we are now able to shed more light on the definition of these parameters.

Lemma 4 *It holds that* $\underline{l}_1(\boldsymbol{r}) = 0$, $\underline{l}_k(\boldsymbol{r}) = (r_k - r_{k-1})^{\frac{1}{k-1}} = \bar{l}_{k-1}(\boldsymbol{r})$, $2 \le k \le c$, $\bar{l}_c(\boldsymbol{r}) = 1$ *for all* $\boldsymbol{r} \in \mathcal{R}_c$.

Proof: See Supplement S.4.

With Lemma 4, we get closer to deriving a manageable expression of the probability function, albeit through an implicit definition of the lower and upper bounds, $\underline{l}_{kj}(\boldsymbol{r})$ and $\bar{l}_{kj}(\boldsymbol{r})$ (refer to Section 3.2). These bounds are defined either such that $\int_{\underline{l}_{kj}(\boldsymbol{r})}^{\bar{l}_{kj}(\boldsymbol{r})} F_\omega\left(\frac{r_k-r_j}{\sum_{m=j}^{k-1} l^m}\right) f_\lambda(l)\, dl = 0$, which does not impact the probability function and can thus be disregarded, or as the intersection points between two curves, which will be our focus (for an illustrative reference, see Figure 2).

By identifying these bounds as intersection points, we establish that for almost every upper bound $\bar{l}_{kj}(\boldsymbol{r})$ (with a single exception as noted in Remark 1), there exists a corresponding lower bound $\underline{l}_{ij}(\boldsymbol{r})$ such that $\bar{l}_{kj}(\boldsymbol{r}) = \underline{l}_{ij}(\boldsymbol{r})$ and $\frac{r_k-r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m} = \frac{r_i-r_j}{\sum_{m=j}^{i-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m}$. Moreover, we observe that small enough variations of the price vector at most change the place where both curves intersect, while they do not change which two curves intersect. Particularly, small enough variations of r_m, $m \notin \{j,k,i\}$, do not change the matching of $\bar{l}_{kj}(\boldsymbol{r})$ and $\underline{l}_{ij}(\boldsymbol{r})$.

Referring to the proof of Lemma 4, we can also add that $\underline{l}_j(\boldsymbol{r}) = \underline{l}_{j,j-1}(\boldsymbol{r})$ and $\underline{l}_{j+1}(\boldsymbol{r}) = \underline{l}_{j+1,j}(\boldsymbol{r})$. Again, small enough variations in the price vector at most change the place of these intersection points.

The following remark summarizes the observations above and will come in handy in the development of optimality conditions.

Remark 2 *It holds that:*

- *For every* $\bar{l}_{kj}(\boldsymbol{r}) \neq \bar{l}_j^{\text{end}}(\boldsymbol{r})$ *there is* $\underline{l}_{ij}(\boldsymbol{r})$ *such that* $\bar{l}_{kj}(\boldsymbol{r}) = \underline{l}_{ij}(\boldsymbol{r})$ *and* $\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m} = \frac{r_i - r_j}{\sum_{m=j}^{i-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m}$. *Moreover,* $\frac{d}{dr_m}\bar{l}_{kj}(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{ij}(\boldsymbol{r})$ for all m *and* $\frac{d}{dr_m}\bar{l}_{kj}(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{ij}(\boldsymbol{r}) = 0$ *for* $m \notin \{j,k,i\}$.
- $\underline{l}_{j+1}(\boldsymbol{r}) = \underline{l}_{j+1,j}(\boldsymbol{r})$ *with* $\frac{r_{j+1} - r_j}{\left(\underline{l}_{j+1}(\boldsymbol{r})\right)^j} = 1$. *Moreover,* $\frac{d}{dr_m}\underline{l}_{j+1}(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{j+1,j}(\boldsymbol{r})$ for all m *and* $\frac{d}{dr_m}\underline{l}_{j+1}(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{j+1,j}(\boldsymbol{r}) = 0$ *for* $m \notin \{j, j+1\}$.
- $\underline{l}_j(\boldsymbol{r}) = \underline{l}_{j,j-1}(\boldsymbol{r})$ *with* $\frac{r_j - r_{j-1}}{\left(\underline{l}_j(\boldsymbol{r})\right)^{j-1}} = 1$. *Moreover,* $\frac{d}{dr_m}\underline{l}_j(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{j,j-1}(\boldsymbol{r})$ for all m *and* $\frac{d}{dr_m}\underline{l}_j(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{j,j-1}(\boldsymbol{r}) = 0$ *for* $m \notin \{j-1, j\}$.

4.3 Optimality conditions

With the previous section, we gathered enough information regarding the probability function to advance to our main goal: optimizing value function (10).

Before we engage the partial differentiation of the value function, we first want to elaborate more on the partial differentiation of probability function (11). The calculation of $\frac{d}{d\,r_i}p_j(\boldsymbol{r})$ varies a little depending on the following three cases: $i > j$, $i < j$, and $i = j$.

Lemma 5 *It holds:*

1. *For* $i > j$, $\frac{d}{d\,r_i}p_j(\boldsymbol{r}) = \int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})} \frac{1}{\sum_{m=j}^{i-1} l^m}\, f_\omega\left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}\right) f_\lambda(l)\, dl$
2. *For* $i < j$, $\frac{d}{d\,r_i}p_j(\boldsymbol{r}) = \int_{\underline{l}_{ji}(\boldsymbol{r})}^{\bar{l}_{ji}(\boldsymbol{r})} \frac{1}{\sum_{m=i}^{j-1} l^m}\, f_\omega\left(\frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m}\right) f_\lambda(l)\, dl$
3. *For* $i = j$, $\frac{d}{d\,r_i}p_j(\boldsymbol{r}) = -\sum_{k=i+1}^{c} \int_{\underline{l}_{ki}(\boldsymbol{r})}^{\bar{l}_{ki}(\boldsymbol{r})} \frac{1}{\sum_{m=i}^{k-1} l^m}\, f_\omega\left(\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m}\right) f_\lambda(l)\, dl -$ $\sum_{k=0}^{i-1} \int_{\underline{l}_{ik}(\boldsymbol{r})}^{\bar{l}_{ik}(\boldsymbol{r})} \frac{1}{\sum_{m=k}^{i-1} l^m}\, f_\omega\left(\frac{r_i - r_k}{\sum_{m=k}^{i-1} l^m}\right) f_\lambda(l) dl$

Proof: See Supplement S.5.

With the additional knowledge about the probability function, we now can turn our focus on the first-order condition. Therefore, we calculate the partial differentiation of value function (10):

$$\frac{d}{d\,r_i}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r})\cdot\left(r_j-\Delta_j V_{t-1}(c)\right)\right)=p_i(\boldsymbol{r})+\sum_{j=1}^{c}\left(\frac{d}{d\,r_i}p_j(\boldsymbol{r})\right)\cdot\left(r_j-\Delta_j V_{t-1}(c)\right)$$

$$=p_i(\boldsymbol{r})+\sum_{j=1}^{i-1}\int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})}\frac{1}{\sum_{m=j}^{i-1}l^m}\,f_\omega\left(\frac{r_i-r_j}{\sum_{m=j}^{i-1}l^m}\right)f_\lambda(l)\,dl\cdot\left(r_j-\Delta_j V_{t-1}(c)\right)$$

$$-\sum_{k=i+1}^{c}\int_{\underline{l}_{ki}(\boldsymbol{r})}^{\bar{l}_{ki}(\boldsymbol{r})}\frac{1}{\sum_{m=i}^{k-1}l^m}\,f_\omega\left(\frac{r_k-r_i}{\sum_{m=i}^{k-1}l^m}\right)f_\lambda(l)\,dl\cdot\left(r_i-\Delta_i V_{t-1}(c)\right)$$

$$-\sum_{k=0}^{i-1}\int_{\underline{l}_{ik}(\boldsymbol{r})}^{\bar{l}_{ik}(\boldsymbol{r})}\frac{1}{\sum_{m=k}^{i-1}l^m}\,f_\omega\left(\frac{r_i-r_k}{\sum_{m=k}^{i-1}l^m}\right)f_\lambda(l)\,dl\cdot\left(r_i-\Delta_i V_{t-1}(c)\right)$$

$$+\sum_{j=i+1}^{c}\int_{\underline{l}_{ji}(\boldsymbol{r})}^{\bar{l}_{ji}(\boldsymbol{r})}\frac{1}{\sum_{m=i}^{j-1}l^m}\,f_\omega\left(\frac{r_j-r_i}{\sum_{m=i}^{j-1}l^m}\right)f_\lambda(l)\,dl\cdot\left(r_j-\Delta_j V_{t-1}(c)\right)$$

Remark 3 *While this expression is still extensive, the level of difficulty strongly depends on the distribution of ω. For example, if ω is uniformly distributed, this expression simplifies to* $\frac{d}{d\,r_i}\left(\sum_{j=1}^{c}p_j(\boldsymbol{r})\cdot\left(r_j-\Delta_j V_{t-1}(c)\right)\right)=2p_i(\boldsymbol{r})-\left(F_\lambda\left(\underline{l}_{i+1}(\boldsymbol{r})\right)-F_\lambda\left(\underline{l}_i(\boldsymbol{r})\right)\right)-\sum_{k=i+1}^{c}\int_{\underline{l}_{ki}(\boldsymbol{r})}^{\bar{l}_{ki}(\boldsymbol{r})}\frac{\Delta_k V_{t-1}(c)-\Delta_i V_{t-1}(c)}{\sum_{m=i}^{k-1}l^m}f_\lambda(l)\,dl+\sum_{k=0}^{i-1}\int_{\underline{l}_{ik}(\boldsymbol{r})}^{\bar{l}_{ik}(\boldsymbol{r})}\frac{\Delta_i V_{t-1}(c)-\Delta_k V_{t-1}(c)}{\sum_{m=k}^{i-1}l^m}f_\lambda(l)\,dl$

Building on the more good-natured expression outlined in Remark 3, we can state the following optimality condition.

Proposition 1 *If* $\omega\sim U[0,1]$*, the optimal solution of (11) meets for every batch size* i *the following condition:*

$p_i(\boldsymbol{r})=\frac{1}{2}\left(F_\lambda\left(\underline{l}_{i+1}(\boldsymbol{r})\right)-F_\lambda\left(\underline{l}_i(\boldsymbol{r})\right)+\sum_{k=i+1}^{c}\int_{\underline{l}_{ki}(\boldsymbol{r})}^{\bar{l}_{ki}(\boldsymbol{r})}\frac{\Delta_k V_{t-1}(c)-\Delta_i V_{t-1}(c)}{\sum_{m=i}^{k-1}l^m}f_\lambda(l)\,dl-\sum_{k=0}^{i-1}\int_{\underline{l}_{ik}(\boldsymbol{r})}^{\bar{l}_{ik}(\boldsymbol{r})}\frac{\Delta_i V_{t-1}(c)-\Delta_k V_{t-1}(c)}{\sum_{m=k}^{i-1}l^m}f_\lambda(l)\,dl\right)$

Proof: See Supplement S.6.

With this optimality condition, we can calculate the optimal probability to sell at least one unit. Thereby, we can make the following observation.

Remark 4 *Let* $\omega\sim U[0,1]$*. For the optimal solution* $\boldsymbol{r}^*$ *of (10), it holds that* $\sum_{i=1}^{c}p_i(\boldsymbol{r}^*)=\frac{1}{2}-\sum_{k=1}^{c}\int_{\underline{l}_{k0}(\boldsymbol{r}^*)}^{\bar{l}_{k0}(\boldsymbol{r}^*)}\frac{\Delta_k V_{t-1}(c)}{\sum_{m=0}^{k-1}l^m}f_\lambda(l)\,dl$*. Thus, the overall selling probability is less than or equal to* 0.5 *and decreasing with opportunity costs.*

States in the last period ($t = 1$) yield no opportunity costs. There, the optimality condition simplifies to $p_i(\boldsymbol{r}) = \frac{1}{2}\left(F_\lambda\left(\underline{l}_{i+1}(\boldsymbol{r})\right) - F_\lambda\left(\underline{l}_i(\boldsymbol{r})\right)\right)$. Moreover, the optimal overall selling probability is 0.5 observable by $\sum_{i=1}^{c} p_i(\boldsymbol{r}) = \frac{1}{2} \cdot \sum_{i=1}^{c}\left(F_\lambda\left(\underline{l}_{i+1}(\boldsymbol{r})\right) - F_\lambda\left(\underline{l}_i(\boldsymbol{r})\right)\right) = \frac{1}{2}\left(F_\lambda(1) - F_\lambda(0)\right) = \frac{1}{2}$.

In a scenario, where also λ is uniformly distributed, the complexity of the customer choice model is further reduced. This, in turn, allows us to derive the following refinement of Proposition 1.

Proposition 2 *Let $\omega, \lambda \sim U[0,1]$. If $\Delta_k V_{t-1}(c) = 0$ for every $k \leq c$, the optimal solution for (10) is an interior point of $\mathcal{R}_c$ and fulfills $p_i(\boldsymbol{r}) = \frac{1}{2}\left(\underline{l}_{i+1}(\boldsymbol{r}) - \underline{l}_i(\boldsymbol{r})\right)$ for every $i \leq c$.*

Proof: To prove that the optimal solution $\boldsymbol{r}^*$ is an interior point, we show $p_k(\boldsymbol{r}^*) \neq 0$ for every k by contradiction. To achieve this, we leverage the observation that $p_k(\boldsymbol{r}) \neq 0$ is equivalent to $\underline{l}_k(\boldsymbol{r}) < \underline{l}_{k+1}(\boldsymbol{r})$, for $\omega, \lambda \sim U[0,1]$.

We assume that for the optimal solution $\boldsymbol{r}^* \in \mathcal{R}_c$ there exists a batch size k with $p_k(\boldsymbol{r}^*) = 0$. In the following, we focus on a single k. However, the technique we use in this proof could be repeatedly applied (with small adjustments) to contradict optimal solutions with several batch sizes k such that $p_k(\boldsymbol{r}^*) = 0$.

In a first step, we modify the optimal solution by slightly decreasing r_k^* to $r_k^* - \epsilon$ with $\epsilon > 0$ small enough. We denote this modified version by $\boldsymbol{r}^{*,\epsilon}$. With $\boldsymbol{r}^{*,\epsilon}$, the probability to sell a batch of size k is positive as $\underline{l}_k(\boldsymbol{r}^{*,\epsilon}) < \underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})$.

<u>$k > 1$</u>:

We recall the argumentation used in Lemma 4 to conclude that $p_k(\boldsymbol{r}^{*,\epsilon}) = \left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right) + \int_{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{l^k}\, dl - \int_{\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}}{l^{k-1}}\, dl$ with $\bar{l}(\boldsymbol{r}^{*,\epsilon}) = \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}} = \frac{r_{k+1}^* - r_k^* + \epsilon}{r_k^* - \epsilon - r_{k-1}^*}$. This $\bar{l}(\boldsymbol{r}^{*,\epsilon})$ exists for every ϵ that is small enough.

Plucking $\boldsymbol{r}^{*,\epsilon}$ into the first partial deviation leads to:

$$\frac{d}{dr_k}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right)\Bigg|_{\boldsymbol{r}=\boldsymbol{r}^{*,\epsilon}} = 2 \cdot p_k(\boldsymbol{r}^{*,\epsilon}) - \left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right)$$

$$= 2 \cdot \left(\left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right) + \int_{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{l^k}\, dl - \int_{\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}}{l^{k-1}}\, dl\right)$$

$$- \left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right)$$

As $1 \geq \frac{\left(r_k^{*,\epsilon}-r_{k-1}^{*,\epsilon}\right)^k}{\left(r_{k+1}^{*,\epsilon}-r_k^{*,\epsilon}\right)^{k-1}} \geq \frac{1}{2}$ for ϵ small enough, it holds that $\left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right) + \int_{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_{k+1}^{*,\epsilon}-r_k^{*,\epsilon}}{l^k}\, dl - \int_{\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_k^{*,\epsilon}-r_{k-1}^{*,\epsilon}}{l^{k-1}}\, dl < \frac{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})-\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}{2}$. Consequently, it holds that $\frac{d}{dr_k}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right)\Big|_{\boldsymbol{r}=\boldsymbol{r}^{*,\epsilon}} < 0$.

Based on the negativity of the partial deviation, the value function is strictly decreasing for every $\boldsymbol{r}^{*,\epsilon}$ (with $\epsilon > 0$ small enough). Thus, in contradiction to the optimality of $\boldsymbol{r}^*$, any $\boldsymbol{r}^{*,\epsilon}$ yields higher expected revenues than $\boldsymbol{r}^*$. A similar argumentation works for r_c if we assume $p_c(\boldsymbol{r}^*) = 0$.

<u>$k = 1$:</u>

Our assumption of $p_1(\boldsymbol{r}^*) = 0$ implies that $r_1^* = r_2^*$. To show the suboptimality of this condition, we need to examine two cases: $r_2^* > \exp(-0.5)$ and $r_2^* \leq \exp(-0.5)$. First, we assume that $r_2^* > \exp(-0.5)$.

By definition of $p_1(\boldsymbol{r}^{*,\epsilon})$ (cf. (11)), $p_1(\boldsymbol{r}^{*,\epsilon}) = (r_2^{*,\epsilon} - r_1^{*,\epsilon}) + \int_{r_2^{*,\epsilon}-r_1^{*,\epsilon}}^{\frac{r_2^{*,\epsilon}-r_1^{*,\epsilon}}{r_1^{*,\epsilon}}} \frac{r_2^{*,\epsilon}-r_1^{*,\epsilon}}{l}\, dl - \int_0^{\frac{r_2^{*,\epsilon}-r_1^{*,\epsilon}}{r_1^{*,\epsilon}}} r_1^{*,\epsilon}\, dl$. Solving the integrals leads to $p_1(\boldsymbol{r}^{*,\epsilon}) = -\log(r_1^{*,\epsilon}) \cdot (r_2^{*,\epsilon} - r_1^{*,\epsilon})$. It holds that $\frac{d}{dr_k}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right)\Big|_{\boldsymbol{r}=\boldsymbol{r}^{*,\epsilon}} < 0$ for $r_1^{*,\epsilon} \in (\exp(-0.5), r_2^{*,\epsilon})$. From here on, the same argumentation as above applies.

Next, we assume that $r_2^{*,\epsilon} \leq \exp(-0.5)$. As every customer with $w \cdot (1 + l) \geq r_2^{*,\epsilon}$ is purchasing at least 2 units, we can conclude that $\sum_{j=1}^{c} p_j(\boldsymbol{r}^{*,\epsilon}) \geq 1 - \int_0^1 \frac{r_2^{*,\epsilon}}{1+l} dl = 1 - r_2^{*,\epsilon} \cdot (\log(2) - \log(1)) \geq 1 - \exp(-0.5) \cdot \log(2) > 0.5$. Since the optimal solution must fulfill the condition $\sum_{j=1}^{c} p_j(\boldsymbol{r}) = 0.5$ (cf. Remark 3), $r_2^{*,\epsilon} \leq \exp(-0.5)$ cannot occur and has no relevance for our proof.

All in all, we have shown that $\boldsymbol{r}$ with r_k such that $p_k(\boldsymbol{r}) = 0$ cannot be an optimal solution. This ultimately leads to the statement that the optimal solution is an interior point of $\mathcal{R}_c$ and has to fulfill the first order condition. □

Although we have developed the optimality conditions, finding all solutions in every state that fulfill these equations is a difficult task. We are facing a system with c nonlinear equations that are additionally plagued by integrals and implicitly defined variables. Most of these difficulties arise from the analytical intractability of the choice model. Thus, online solving (10) to optimality in every state is out of scope. Consequently, this necessitates the adoption of heuristic methods. Before exploring our heuristic approaches, we first consider theoretical performance guarantees and establish methodologies to ensure these guarantees within our algorithms.

4.4 Fluid approximation and asymptotic optimality

Asymptotic optimality is a desirable property for a heuristic solution method to ensure theoretical performance guarantees. In literature (see, e.g., Maglaras & Meissner, 2006), this is often ensured by the solution of a deterministic fluid model.

This section is built as follows: We start with formulating the deterministic fluid model and discussing the asymptotic optimality of its solution. The analytical complexity introduced by our choice model necessitates the adoption of a simplified, alternative model that not only provides an upper bound to the fluid model but also is less complex. We culminate by demonstrating that the optimal solution derived from the modified model retains its optimality in the original fluid model, achieving identical objective outcomes.

4.4.1. Original fluid approximation

In the fluid model, we assume that capacity is continuously divisible, and demand is no longer given by a stochastic progress. It is rather deterministically determined by its rate and, thus, we assume that customers purchase continuous fractions of batches.

We denote deterministic continuous demand for batch j at time t by $p_j^d(\boldsymbol{r}_t)$ with $p_j^d(\boldsymbol{r}_t) = p_j(\boldsymbol{r}_t)$ and capacity is depleted by $\sum_{j=1}^{C} j \cdot p_j^d(\boldsymbol{r}_t)$ at time t. The fluid model is given by:

$$\max_{\boldsymbol{r}_t \in \mathcal{R}_c \, \forall t} \left\{ \sum_{t=1}^{T} \sum_{j=1}^{C} r_{jt} \cdot p_j^d(\boldsymbol{r}_t) \, : \, \sum_{t=1}^{T} \sum_{j=1}^{C} j \cdot p_j^d(\boldsymbol{r}_t) \leq C \; and \; \boldsymbol{r}_t \geq 0 \right\}$$

With a time-homogeneous demand function, one can easily verify that the optimization problem above is equivalent to:

$$\max_{\boldsymbol{r} \in \mathcal{R}_c} \left\{ T \cdot \sum_{j=1}^{C} r_j \cdot p_j^d(\boldsymbol{r}) \, : \, \sum_{j=1}^{C} j \cdot p_j^d(\boldsymbol{r}) \leq \frac{C}{T} \right\} \tag{12}$$

With this equality, the optimization problem simplifies to a static pricing policy that proves optimal for the fluid model. Notably, the fluid model (12) can be brought into the form of the multiproduct fluid model examined by Maglaras and Meissner (2006). Similarly, their proof regarding asymptotic optimality of the fluid approximation's solution also holds for the static pricing policy received by (12). Applying this static policy in our dynamic setting is therefore asymptotically optimal in a regime where demand and capacity grow proportionally large. This definition is called the first-order asymptotic optimality criterion by Gallego and van Ryzin (1997) and Cooper (2002). Moreover, Maglaras and Meissner (2006) show that resolving the fluid policy throughout the selling horizon is again asymptotically optimal.

Remark 5 *The optimal solution of the unrestricted version of (12) is equivalent to the optimal solution of the dynamic model (10) in states* $t = 1$. *Thereby, for* $\omega, \lambda \sim U[0,1]$,*, Proposition 2 also defines the optimal solution for this case. If this solution does not fulfill* $\sum_{j=1}^{C} j \cdot p_j^d(\boldsymbol{r}) \leq \frac{C}{T}$, *we can still leverage* $\frac{d}{d\,r_i}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right) = p_i(\boldsymbol{r}) - \frac{1}{2}\left(\underline{l}_{i+1}(\boldsymbol{r}) - \underline{l}_i(\boldsymbol{r})\right)$ *to find the optimal solution.*

Solving the fluid approximation is less complex than solving the dynamic program since it involves only one stationary optimization instead of up to $C \cdot T$ different state-wise optimizations. Furthermore, in this stationary optimization, we do not have to account for opportunity costs, which simplifies the optimality condition. However, this optimality condition still involves a system of c non-linear, dependent equations that have to be solved, motivating our search for further simplifications.

4.4.2. Modified fluid model

The existing fluid model encounters analytical challenges due to the complexity introduced by our choice model. To address this, we introduce a modified fluid model that simplifies the choice model, thus facilitating a more manageable analysis.

Instead of calculating the probability that $u_k(\boldsymbol{r}) = \max_{j=0,\dots,C}\{u_j(\boldsymbol{r})\}$, we now calculate the probability that $u_k(\boldsymbol{r}) = \max_{j=0,k-1,k,k+1}\{u_j(\boldsymbol{r})\}$. Thereby, we reduce competition between available options and, thus, only compare three instead of $c-1$ options. We denote demand by this modified choice model as $p_j^{d,a}(\boldsymbol{r})$ and write $p_j^{d,a}(\boldsymbol{r}) = \mathbb{P}\left(u_j(\boldsymbol{r}) = \max_{k=0,j-1,j,j+1}\{u_k(\boldsymbol{r})\}\right) \geq p_j^d(\boldsymbol{r})$. Technically, through the alternation, the choice model is no longer a choice model. By summing up all demand rates, we get a value greater than or equal to 1, i.e. $\sum_{j=0}^{C} p_j^{d,a}(\boldsymbol{r}) \geq \sum_{j=0}^{C} p_j^d(\boldsymbol{r}) = 1$, which does not satisfy one of the core assumptions regarding choice models. The modified fluid model is given by:

$$\max_{\boldsymbol{r}\in\mathcal{R}_c}\left\{T \cdot \sum_{j=1}^{C} r_j \cdot p_j^{d,a}(\boldsymbol{r}) : \sum_{j=1}^{C} j \cdot p_j^{d,a}(\boldsymbol{r}) \leq \frac{C}{T}\right\} \tag{13}$$

The optimal solution of this modified model is an upper bound to (12). This can be verified as follows: The optimal solution $\boldsymbol{r}^*$ of (12) is either feasible for (13) or $\sum_{j=1}^{C} j \cdot p_j^{d,a}(\boldsymbol{r}^*) > \frac{C}{T}$. If it is feasible, the statement is obviously true considering $p_j^{d,a}(\boldsymbol{r}^*) \geq p_j^d(\boldsymbol{r}^*)$. In the letter case, we can increase $\boldsymbol{r}^*$ in such a way to $\boldsymbol{r}^{**}$ that $p_j^{d,a}(\boldsymbol{r}^{**}) = p_j^d(\boldsymbol{r}^*)$. Thereby, we impose the same demands with higher prices.

Similar considerations as in Section 4.3 lead to the following remark.

Remark 6 *With $p_j^{d,a}(\boldsymbol{r})$, the statement of Remark 5 holds analogously.*

This modification significantly reduces complexity in our choice model (see Supplement S.8 for a calculation of $p_j^{d,a}(\boldsymbol{r})$ for a scenario where $\omega, \lambda \sim U[0,1]$). This adjustment enables the efficient numerical determination of the optimal solution $\boldsymbol{r}^*$ for the modified fluid model, under scenarios where parameters $\omega, \lambda \sim U[0,1]$. Identifying $\boldsymbol{r}^*$ is merely the initial phase; the subsequent and critical phase involves validating its optimality across both the modified and original fluid models. Utilizing the fact that the modified model constitutes an upper bound to the original model, it suffices to verify that $\boldsymbol{r}^*$ is feasible for the original model and results in the same objective value. To facilitate this verification, we propose an approach that avoids the exhaustive checking whether $p_j^{d,a}(\boldsymbol{r}^*) = p_j^d(\boldsymbol{r}^*)$ for every j.

Instead, we introduce a set of straightforward and verifiable conditions, designed to simplify the process of proving $\boldsymbol{r}^*$'s optimality for the original fluid model (12).

Proposition 3 *If $\boldsymbol{r}$ meets the conditions 1.-3., then $p_j^{d,a}(\boldsymbol{r}) = p_j^d(\boldsymbol{r})$ for every j:*

1. $\bar{l}_j^{end}(\boldsymbol{r}) \leq \frac{r_{j+1}-r_j}{r_j-r_{j-1}}$ *for every j, with $1 < j < c$,*
2. $\bar{l}_j^{end}(\boldsymbol{r}) \leq \bar{l}_{j+1}^{end}(\boldsymbol{r})$ *for every j, with $1 \leq j < c$,*
3. $\sum_{k=0}^{j-1} \left(\bar{l}_j^{end}(\boldsymbol{r})\right)^{k-j} = \frac{r_j}{r_{j+1}-r_j}$.

Proof: See Supplement S.7.

In our numerical study, we calculated the optimal solution for the modified fluid model for every combination of $T \leq 40$ and $C \leq 120$, under the assumption that $\omega, \lambda \sim U[0,1]$. The conditions of Proposition 3 were always fulfilled. Therefore, in these instances, this solution could be used as a static pricing policy that is asymptotically optimal in our dynamic setting. The same holds for a policy that periodically solves the modified fluid model with actualized capacity and time-to-go. Moreover, any policy that yields higher expected revenues at every stage of the optimization problem is again asymptotically optimal. This insight lays the foundation for ensuring asymptotic optimality of our heuristics.

5 Asymptotically optimal heuristics

In this section, we construct three heuristics to solve the problem regardless of its analytical intractability. Two of these approaches are based on the results of Schur (2024) and use the optimal solution in a setting where the firm has access to customers' private information, i.e., their base willingness-to-pay and their consumption indicator. respectively. The third approach can be described as a decomposition in units. Thereby, we allow customers to buy the jth unit of the product without buying the units $1, 2, \ldots, j-1$. Even though this does not reflect reality, it constitutes an easy to solve optimization problem, enabling us to devise batch pricing strategies for the original problem based on the solutions obtained.

Reflecting on the fluid model's asymptotic optimality discussed previously, we aim to preserve this property within our heuristic frameworks. To this end, we evaluate the performance of each heuristic against the fluid model's solution in every state, adopting the superior option. This systematic comparison ensures that our heuristics not only address the problem's intractability but also maintain theoretical performance integrity.

5.1 Approaches 1 and 2: Expected optimal batch prices

With the results of Schur (2024), we can efficiently compute realization-dependent optimal batch prices $r_{jt}(c|w)$ and $r_{jt}(c|l)$ for realization w and l, respectively. Technically, these realization-dependent batch prices are themselves random variables, raising the following idea: By calculating the expected value of these realization-dependent optimal batch prices, we obtain a reliable estimate for the optimal solution of optimization problem (8).

Both approaches follow the same idea, differing only in the determination of the realization-dependent optimal batch prices: $r_{jt}(c|w)$ for Approach 1 and $r_{jt}(c|l)$ for Approach 2. Beyond this distinction, both approaches follow the same subsequent steps. Consequently, we will explain the remaining steps without distinguishing between both approaches and write $r_{jt}(c|x)$ instead of $r_{jt}(c|w)$ and $r_{jt}(c|l)$ to denote the realization-dependent optimal batch prices.

This framework is applicable across a wide spectrum of distribution functions, including, but not limited to, uniform, triangular, normal, exponential, Weibull, Gumbel, and gamma distributions, along with their truncated versions, albeit with some constraints on parameter selections (as detailed in Schur, 2024). Notably, for scenarios where $\omega \sim U[0,1]$, Approach 2 provides a closed-form expression for the realization-dependent optimal batch prices: $r_{jt}(c|l) = \frac{1}{2}\left(\sum_{k=0}^{j-1} l^k + \Delta_j V_{t-1}^E(c)\right)$, with $j \leq N_t(c|l) = \max_{j=1,\dots,c} \{j: \Delta_1 V_{t-1}^E(c-j+1) < l^{j-1}\}$.

Building on these realizations-dependent optimal batch prices, we start our heuristics by building expected optimal batch prices:

$$r_{jt}^E(c) = \frac{\int_0^1 r_{jt}(c|x) \cdot 1_{\{j<N_t(c|x)\}} f(x)\, dx}{\int_0^1 1_{\{j<N_t(c|x)\}} f(x)\, dx} \qquad \text{for } j = 1, \dots, c, \tag{14}$$

More precisely, this formulation results in conditional expected optimal batch prices, where we only take realizations of λ and ω into account that lead to possible economic sales, i.e. $r_{j+1,t}(c|\cdot) - r_{jt}(c|\cdot) \geq \Delta_1 V_{t-1}^E(c+1-j)$. Other realizations are economically irrelevant and can distort results, given the lack of a clear pricing strategy in these cases. Thus, these events where we refrain from selling are not used to find overall good batch prices.

Except for the scenario where we have a closed-form expression of $r_{jt}(c|l)$, we resort to numerically calculating $r_{jt}(c|x)$ for as many realizations x as possible to accurately derive $r_{jt}^E(c)$. Finally, we compare expected revenue-to-go derived by expected optimal batch prices $r_{jt}^E(c)$ and the solution of the fluid approximation $r_{jt}^{FA}(c)$:

$$V_t^E(c) = \max_{\boldsymbol{r} \in \{\boldsymbol{r}_t^E(c), \boldsymbol{r}_t^{FA}(c)\}} \left\{\sum_{j=1}^c p_j(\boldsymbol{r}) \cdot \left(r_j + V_{t-1}^E(c-j)\right) + \left(1 - \sum_{j=1}^c p_j(\boldsymbol{r})\right) \cdot V_{t-1}^E(c)\right\}, \tag{15}$$

with the same boundary conditions as the original problem (8), applying our original probability function $p_j(\boldsymbol{r})$. The better-performing batch prices are then adopted by our heuristics.

Remark 7 *By using suboptimal batch prices, we get a lower bound to optimization problem (8). Thus, it holds that* $V_t(c) \geq V_t^E(c)$ *for every* (t, c).

We sum up the first two heuristics that are based upon the idea of expected optimal batch prices by the following pseudo code:

Pseudo code Approaches 1 and 2: Expected optimal batch prices

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Input: time horizon T, starting stock C

Output: batch prices for every batch size j and state (t, c): $r_{jt}^E(c)$

---

1. Initialize value functions $V_0^E(c) = 0$ for every $c = 0, 1, \dots, C$ and $V_t^E(0) = 0$ for every $t = 0, 1, \dots, T$ ▷ initialize boundary conditions

2. For $t = 1, 2, \dots, T$ **do** ▷ loop over time horizon

2.1. For $c = 1, 2, \dots, C$ **do** ▷ loop over capacity

2.1.1. Calculate $r_{jt}^E(c)$ for every $j = 1, 2, \dots, c$ ▷ calculate expected optimal batch prices

2.1.2. Calculate $V_t^E(c)$ with (15) ▷ calculate expected revenue-to-go

2.1.3. Save maximizing price vector of (15) as asymptotically optimal solution

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5.2 Approach 3: Decomposition in units

Our next algorithm employs a decomposition strategy. The basic idea is that customers have the flexibility to buy the jth unit of the product even though they might not buy units 1 to $j - 1$. As every unit of the product is the same, there is no distinction between the $1st$, 2nd or jth unit other than the number customers have already in their basket. Thus, this decomposition is merely theoretical without having immediate practical applicability. However, it results in a greatly simplified optimization problem. A hypothetical customer now faces c distinct binary decisions instead of one decision with $c + 1$ options. This, in turn, enables us to solve c distinct and rather simple independent optimization problems instead of one complex problem. With this simplification, we can derive batch prices that can be used as practical estimations for the optimal batch prices to optimization problem (8).

To consider this decomposition, we must change the customer choice model. Customers still strive to maximize their utility. But, instead of purchasing j units if and only if $u_j(\boldsymbol{r}) = \max_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}$ with $u_0(\boldsymbol{r}) = 0$ denoting the no-purchase option, they decide for every single unit whether they want to purchase it or not. This decision is based upon whether the additional willingness-to-pay for the jth unit is at least as high as the additional price the customer has to pay, i.e. $X_j - X_{j-1} = \omega \cdot (\lambda)^{j-1} \geq r_j - r_{j-1}$. If the customers decide to purchase the jth unit, they must pay $r_j - r_{j-1}$. For example, for given batch prices, a customer might only be willing to purchase the second and fourth unit due to the willingness-to-pay curve. In this case, the customer pays $(r_2 - r_1) + (r_4 - r_3)$ to get 2 units of the product in total.

The decomposition approach reduces the complexity of the choice model. The model itself becomes easier as the decision between several options is broken down to several binary independent decisions. This method avoids the need to determine a single price vector that encompasses all batch prices and to predict the customer's response to this vector comprehensively. Instead, the firm's pricing strategy is divided into c separate decisions, each focused on setting the price for the jth unit, while considering the probability of its purchase (which happens solely based on its price). Therefore, the decision variable becomes the price of the jth unit, $r^{D,j} = r_j - r_{j-1}$. The optimization problem we are now focusing on is given by:

$$\sum_{j=1}^{c} \left(\max_{r^{D,j} \geq 0} \left\{ p^{D,j}\left(r^{D,j}\right) \cdot \left(r^{D,j} - \Delta_1 V_{t-1}^{D}\left(c-(j-1)\right)\right) \right\} \right) + V_{t-1}^{D}(c), \tag{16}$$

with $p^{D,j}\left(r^{D,j}\right)$ denoting the probability that a customer is purchasing the jth unit of the product and $V_{t-1}^{D}(c)$ denoting the expected revenue-to-go.

Probability $p^{D,j}\left(r^{D,j}\right)$ only depends on $r^{D,j}$ and can be calculated by $p^{D,j}\left(r^{D,j}\right) = \int_0^1 \int_0^1 1_{\{w \cdot (l)^{j-1} \geq r^{D,j}\}}(w,l)\, f_\omega(w) f_\lambda(l)\, dw\, dl = \int_0^1 1 - F_\omega\left(\frac{r^{D,j}}{l^{j-1}}\right) f_\lambda(l) dl$. Optimization problem (16), which is the sum of c singleunit dynamic pricing problems, greatly simplifies the search for the optimal solution. With $r_t^{D,j}(c)$ denoting the optimal solution to (16), we can use $r_{jt}^{D}(c) = \sum_{i=1}^{j} r_t^{D,i}(c)$ as a practical estimation of the optimal batch prices for the original optimization problem (8). Lastly, applying our original probability function $p_j(\boldsymbol{r})$, we evaluate expected revenue-to-go derived by $r_{jt}^{D}(c)$ and by the solution of the fluid approximation $r_{jt}^{FA}(c)$:

$$V_t^D(c) = \max_{\boldsymbol{r} \in \{\boldsymbol{r}_t^D(c), \boldsymbol{r}_t^{FA}(c)\}} \left\{ \sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j + V_{t-1}^{D}(c-j)\right) + \left(1 - \sum_{j=1}^{c} p_j(\boldsymbol{r})\right) \cdot V_{t-1}^{D}(c) \right\}. \tag{17}$$

The boundary conditions are again given by $V_0^D(c) = 0$ for $c \geq 0$ and $V_t^D(0) = 0$ for $t \geq 0$.

Remark 8 *By using suboptimal batch prices, we get a lower bound to optimization problem (8). Thus, it holds that $V_t(c) \geq V_t^D(c)$ for every (t,c).*

The heuristic can be summed up by the following pseudo code:

Pseudo code Approach 3: Decomposition of units

Input: time horizon T, starting stock C

Output: batch prices for every batch size j and state (t,c): $r_{jt}^{D}(c)$

1. Initialize value functions $V_0^D(c) = 0$ for every $c = 0, 1, \dots, C$ and $V_t^D(0) = 0$ for every $t = 0, 1, \dots, T$ ▷ initialize boundary conditions
2. **For** $t = 1, 2, \dots, T$ **do** ▷ loop over time horizon

2.1. **For** $c = 1, 2, \dots, C$ **do** ▷ loop over capacity

2.1.1. Solve optimization problem (16) to get $r_t^{D,j}(c)$ for every $j = 1, 2, \ldots, c$ ▷ calculate optimal price for the jth unit for every j

2.1.2. Calculate $r_{jt}^{D}(c) = \sum_{i=1}^{j} r_t^{D,i}(c)$ for every $j = 1, 2, \ldots, c$ ▷ calculate proxy for optimal batch prices

2.1.3. Calculate $V_{t-1}^{D}(c)$ with (17) ▷ calculate expected revenue-to-go

2.1.4. Save maximizing price vector of (17) as asymptotically optimal solution

6 Numerical studies

In this section, we examine the performance of all heuristics outlined in the previous section. Nevertheless, we also implemented and simulated the special case *PI* where the firm can observe next customers' consumption indicator (refer to Schur, 2024). Thereby, we get an upper bound for our unknown optimal solution and can compare our heuristics against it.

To evaluate our heuristics, we implemented the following mechanisms:

- $E(\lambda)$ and $E(\omega)$ are mechanisms to approximately solve (8) based on the idea of expected optimal batch prices (Section 5.1).
- *D* is a mechanism to approximately solve (8) based on a decomposition in units (Section 5.2).
- *S* is a mechanism that solves a standard singleunit dynamic pricing problem and, thereby, ignores the fact that customers might be willing to purchase more than just one unit. As a singleunit dynamic pricing procedure results in a price for only one unit of the product, we then extend it in a linear manner to get batch prices.
- *L* is a mechanism that solves (8) numerically but with an additional constraint that restricts the batch prices to a linear pricing scheme, i.e. $r_j = j \cdot r_1$ for every j. Thus, the optimization problem simplifies as we only have one decision variable.
- *PI* is a mechanism where we observe customers' consumption indicator before quoting batch prices; the corresponding optimization model is given in Schur (2024). This mechanism sets optimal batch prices in a setting where the firm has additional information, and thus, yields an upper bound to our setting.

Besides our heuristics, we have chosen *S* and *L* for our numerical studies, as they might be applied the most in practice. Mechanism *S* is conducting standard dynamic pricing and, thus, ignoring multiunit demand. Firms recognizing that customers may purchase more than a single unit are likely to adopt mechanism *L,* as this is the obvious choice without specialized optimization problems that merely exist in literature. The last mechanism, denoted as *PI*, constitutes an upper bound of the (unknown) value of the objective function (8) and thereby provides a benchmark for our heuristics. However, we should keep in mind that *PI* determines the revenue that could be earned if the firm has additional information about customers' preferences, and thus, shows the inherent advantage of this special case in comparison to our case.

Every mechanism provides a policy that contains batch prices. To evaluate the mechanisms, we perform simulation studies We choose the setting for our study similar to Gallego et al. (2020), resulting in the choice of $T = 1, \dots, 40$, $C = 1, \dots, 120$, and the uniform distribution of ω and λ. We generated 10,000 customer streams in advance and applied every policy we derived from the mechanisms to these streams separately. One simulation run consists of a complete sales process containing T specific customers and a batch price quoted in every period (depending on the capacity left) according to the mechanism investigated. After observing the decision of the current customer, a new batch price is set for the next customer. Repeating this procedure until the end of the selling horizon leads to a total revenue for this simulation run. By averaging the total revenues from 10,000 simulation runs, we obtain a mean revenue for each mechanism. Since all mechanisms are evaluated using the same 10,000 customer streams, their simulated revenues are directly comparable.

In the following, we show simulated revenues of all mechanism and compare them to the upper bound received from special case *PI*. Thereby, we can observe that heuristics *D* and $E(\omega)$ are resulting in the highest revenues, while $E(\lambda)$ is slightly behind. We then examine the selling strategies of all heuristics by analyzing simulation results. Moreover, we discuss several evolutions of batch prices of both best performing heuristics *D* and $E(\omega)$. As we thereby observed a piecewise-linear pattern in batch prices, we finally conducted another numerical study to evaluate a piecewise-linear pricing scheme. This study shows that such a pricing scheme is well performing and might be an easy to communicate alternative.

6.1 Comparison in terms of revenue earned

In this section, we show the simulated revenues of the first five mechanisms for the investigated settings. To shorten tables and give a more lucid overview of the study, we provide only a subset of the studied settings in the following table.

Table 1: Revenues for $\boldsymbol{C \leq 120, T = 40}$

$\boldsymbol{T = 40}$	$E(\lambda)$	$E(\omega)$	D	S	L
$\boldsymbol{C = 1}$	0.91	0.91	0.91	0.91	0.91
$\boldsymbol{C = 30}$	16.18	16.23	16.27	15.12	15.74
$\boldsymbol{C = 60}$	23.85	24.07	24.12	22.16	22.86
$\boldsymbol{C = 90}$	28.66	29.03	29.09	25.24	27.32
$\boldsymbol{C = 120}$	32.01	32.51	32.55	27.02	30.43

Table 1 shows simulated revenues of all mechanisms with $C \in \{1, 30, 60, 90, 120\}$. For $C = 1$, every mechanism performs identical as there is not enough stock to sell more than one unit, and thus, the consideration of multiunit purchases is not possible. As we can see, the developed heuristics (mechanisms $E(\cdot)$ and D) are also able to calculate the optimal price for the singleunit case. The more capacity, the higher is the importance of attending customers' demand for more than one unit. This can be seen by comparing mechanisms S and L where the only difference between those mechanisms is that L is aware of customers' multiunit demand while S is ignoring it. Except $C = 1$, L is outperforming S in every setting. This holds all the same for the cases we did not incorporate in this paper. On the other hand, L is dominated by our heuristics. Even though $E(\cdot)$ and D are only heuristics while L numerically finds optimal linear batch prices, the advantage of setting nonlinear batch prices (like $E(\cdot)$ and D do) is overcompensating the incapability to solve optimization problem (8) analytically. Finally, comparing our heuristics to each other, we can observe that $E(\lambda)$ results in the lowest expected revenues while $E(\omega)$ and D are performing nearly identical.

Applying $E(\cdot)$ and D is clearly beneficial in comparison to L and even more so to S. As we do not know the optimal solution to optimization problem (8), we cannot test these mechanisms against it. However, to get a feeling for the performance of all mechanisms, we show the simulated revenues relative to an upper bound. Therefore, we additionally simulated revenue for a policy where we were able to observe the realization of the random consumption indicator λ of an arriving customer before quoting the optimal batch prices (cf. Schur, 2024). We then divided simulated revenues received from each mechanism by the simulated upper bound received from PI and, thereby, get the percentage every mechanism obtains. These percentages are shown in the following figure. In Table 1, we have seen that $E(\omega)$ and D are performing nearly identical. Thus, we opted to only show the line of D representing both mechanisms. We used dotted lines to separate mechanisms that are quoting linear batch prices from the other mechanisms that are following a nonlinear pricing scheme. We let the ordinate start at 0.5 to demonstrate differences between the curves more clearly.

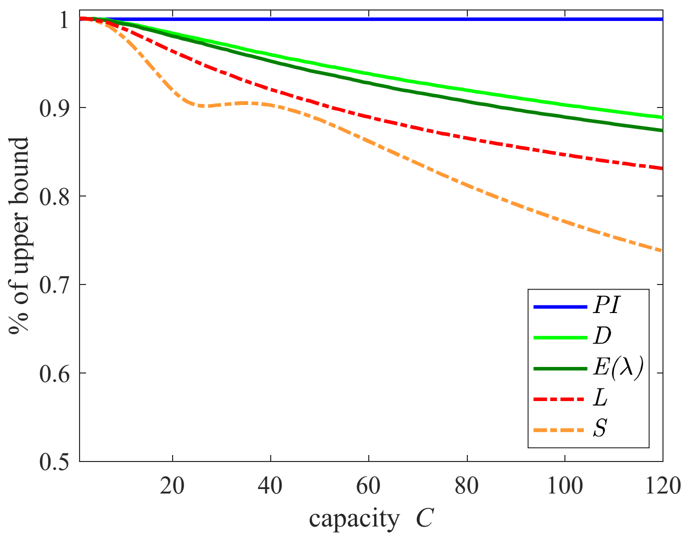

Figure 3: Performance of all mechanism relative to an upper bound for $C \leq 120, T = 40$

Figure 3 shows the same order in the mechanisms for every capacity like Table 1, i.e. $D \geq E \geq L \geq S$. The gap between the upper bound and any other mechanism is increasing in capacity. Probably, this is due to the increasing advantage of knowing the consumption indicator in the upper bound mechanism for higher C. Through full information about customer's consumption indicator and the ability to adapt batch prices to this knowledge, the revenue earned ought to be higher than the (unknown) value of the objective function from optimization problem (8) by a fair amount. This is particularly true whenever there is enough capacity to sell several units to every customer. Thus, we do not assume the gap between upper bound and both heuristics to widen solely because *D* and $E(\cdot)$ are performing less good for higher capacity. Another indication for this assumption is that *S*, the mechanism where multiunit demand is ignored, is decreasing at a high pace while *D*, *E*, and *L* are becoming increasingly stable at the end of the curve.

Table 1 already revealed that *D* and $E(\omega)$ are performing nearly identical even though the idea behind both heuristics is different. However, $E(\lambda)$ is not so far away from these two mechanisms. This could be an indication that all heuristics (especially *D* and $E(\omega)$) are (strongly) bounded by the optimal solution of optimization problem (8) and, thus, are a good approximation. Finally, we note that our heuristics are performing remarkably better (with a small plus for *D*) than the mechanisms that are most likely applied in practice, *L* and *S*. Overall, *D* and $E(\omega)$ are outperforming $E(\lambda)$ particularly for higher capacities.

6.2 Selling strategies of the heuristics

As our heuristics are performing similarly, we want to analyze their strategies. Therefore, we have a closer look to the simulated mean revenue and mean purchases at every point in time. We perform this (and any following) analysis on a smaller setting, to enhance the visibility of effects, and start the observation in state $(T, C) = (10, 20)$. In the following figures, we track purchases and average the resulting revenues at every period and over all customer streams.

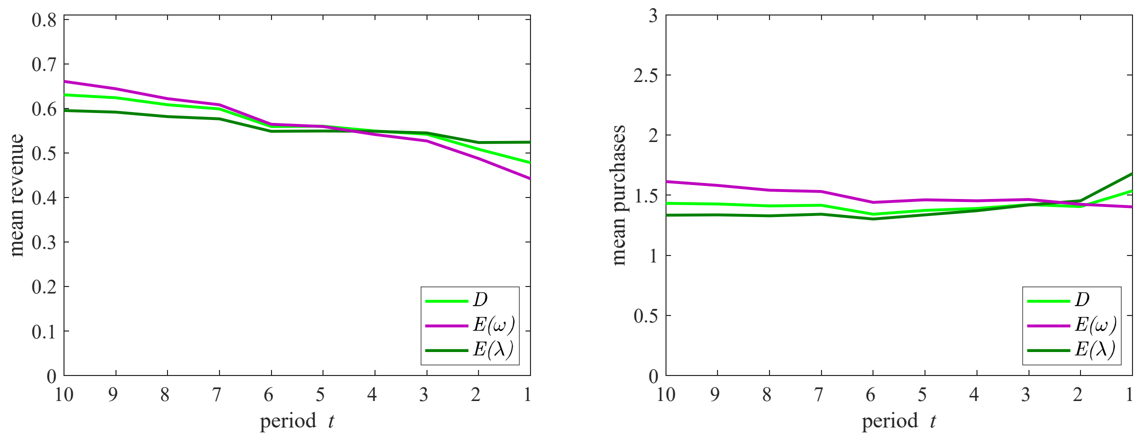

Figure 4: Mean revenues (left) and mean purchases (right) at every point in time, $t = 10, 9, \ldots, 1$ with $C = 20$

Although D, $E(\omega)$, and $E(\lambda)$ are performing similarly overall, the strategies seem to be quite different. Figure 4 shows that $E(\lambda)$ is superior to $E(\omega)$ in terms of mean revenue at the end of the selling horizon, while the opposite is true at the beginning. The line representing the strategy of D is running in between of the lines representing $E(\omega)$ and $E(\lambda)$. Mean revenues of each heuristic are decreasing over time. While $E(\lambda)$ performs quite stable throughout the whole selling process, the curve representing $E(\omega)$ is declining at a more pronounced rate.

Mean purchases are around 1.5 for every mechanism. The curve resulting from $E(\omega)$ is declining over time, while the curves of the other two mechanisms are quite stable with a sharp, single uptick at the end of the horizon. Over the whole horizon, $E(\omega)$ is selling more units than the other two heuristics. D and $E(\lambda)$ are selling similar quantities, with D being slightly above $E(\lambda)$ overall. This suggests that $E(\omega)$ is pricing units lower than the other two mechanisms.

6.3 Pricing path of D and $E(\omega)$

We have already seen in Table 1 that D and $E(\omega)$ are resulting in the highest revenues. By examining pricing paths received by these heuristics, we get insights about the structure and behavior of overall well-performing pricing policies. In the figures we present in this section, the jth curve (counting from the bottom) corresponds to the batch price of j units. The starting inventory is $C = 20$ units at period $T = 10$.

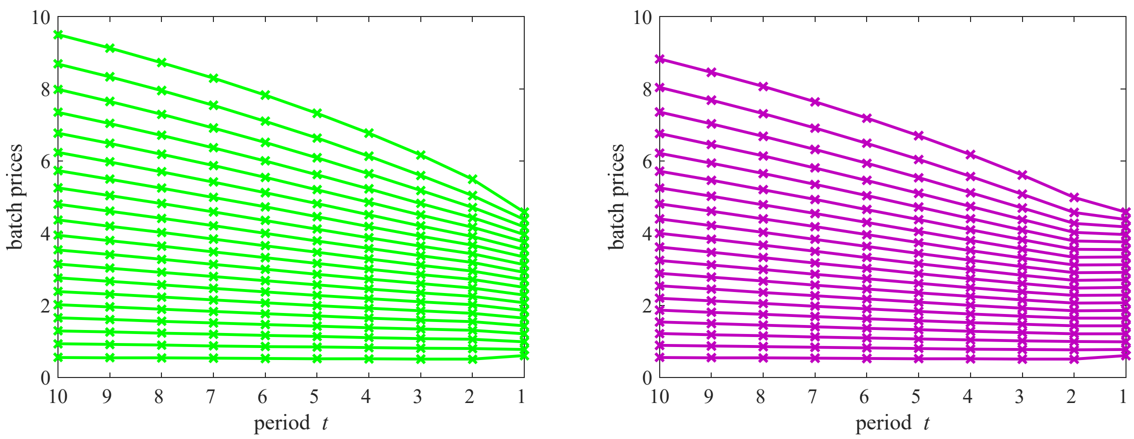

Figure 5: Evolution of batch prices without a purchase for D (left) and $E(\omega)$ (right) over $t = 10, 9, \dots, 1$ and $C = 20$

In Figure 5, we have depicted the case where no customer wants to purchase anything. Over the entire selling horizon, mechanism D quotes higher prices than mechanism $E(\omega)$. Additionally, prices from both mechanisms are decreasing in time with one exception: some batch prices in $t = 1$ are higher than those in $t = 2$. This structural break is a result of enforcing asymptotic optimality. In $t = 1$, every heuristic applies the solution of the fluid approximation $\boldsymbol{r}_t^{FA}(c)$ as it is optimal in the last period (refer to Remark 4). For $t \geq 2$, the maximization picks $\boldsymbol{r}_t^{D}(c)$ and $\boldsymbol{r}_t^{E(\omega)}(c)$ as these prices are resulting in higher revenues than $\boldsymbol{r}_t^{FA}(c)$. Despite the distinct mechanisms employed to calculate $\boldsymbol{r}_t^{D}(c)$ and

$\boldsymbol{r}_t^{E(\omega)}(c)$, the resulting curves have a similar structure. This indicates an underlying structure good policies have in common.

Batch prices for small numbers of units are virtually linear in batch size (apart from the first unit, the second, third, etc. units cost nearly the same). For large numbers of units, batch prices are convexly increasing in batch size. With a concavely increasing willingness-to-pay curve, these pricing schemes automatically prevent selling a large batch or even the whole stock ($C = 20$) to only one customer. Finally, it is notable that prices for small batches merely change over time whereas prices for big batches noticeably decrease.

In Figure 4, we have seen that both heuristics result in selling processes where a customer on average purchases approximately 1.5 units. Therefore, we also want to examine the evolution of batch prices in a scenario where alternately two units and one unit are sold, starting with a purchase of 2 units in $t = 10$. As the firm cannot offer batches that are not covered by capacity any longer, most of the curves are terminated during the selling horizon.

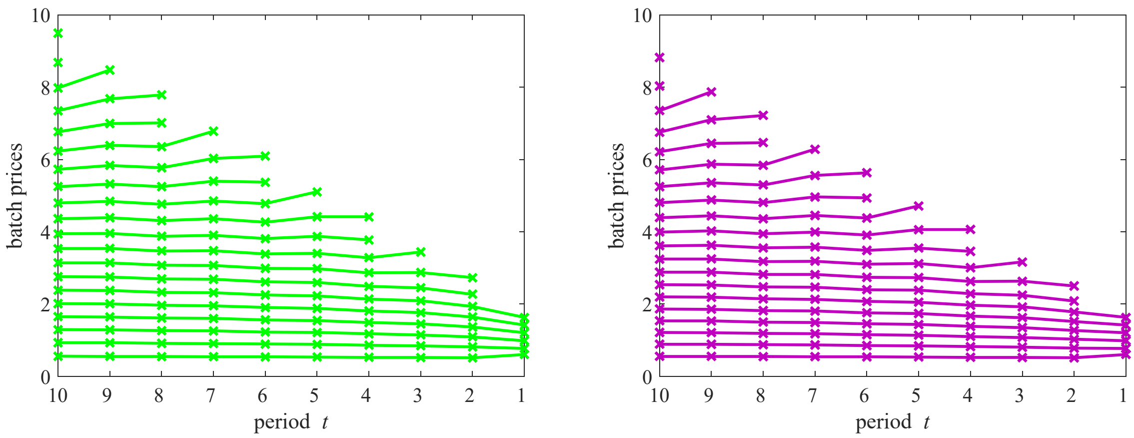

Figure 6: Evolution of batch prices with purchases at every period for D (left) and E (right) over $t = 10, 9, \ldots, 1$ and $C = 20$

The pattern the curves draw looks nearly the same for both heuristics. Batch prices obtained by $E(\omega)$ are lower than those obtained by *D*. The gaps between batch prices are nearly same-sized for smaller batches (again, starting with the two-unit batch) and are increasing for bigger batches. After selling, batch prices for bigger batches increase. This effect is more pronounced after selling two units in comparison to selling one unit. It is a well-observed pricing behavior in (standard) dynamic pricing that prices increase after a sale took place. However, this only holds partially in our multiunit setting as prices for small batches usually decrease slow and steady over the selling horizon.

To sum up, we have seen two (unusual) effects in Figures 5 and 6: First, prices for small (in comparison to remaining stock) batches are decreasing over time regardless whether a sell takes place or not. This holds for the discussed settings with a reasonably large stock, i.e. $\frac{C}{T} = 2$. For a small stock, i.e. $\frac{T}{C} = 2$, prices for small batches were not always decreasing from one period to the next. To shorten this paper, we excluded the corresponding figures. Second, prices for small and medium batches are increasing approximately linearly in size – starting with the two-unit batch.

6.4 Piecewise-linear pricing

Section 6.3 suggests that the marginal prices of additional units ($j \geq 2$) are nearly constant. To evaluate the loss in revenue if we enforce a piecewise-linear pricing scheme, we implemented mechanism *PL*:

- PL is a mechanism that solves (8) numerically but with an additional constraint that restricts the batch prices to a piecewise-linear pricing scheme with r_1 and $r_j = (j-1) \cdot (r_2 - r_1) + r_1$ for $j \geq 2$.

This pricing scheme inherits an easy structure, and thus, could be easily applied in practice because a firm has only to quote a price for the first and a price for subsequent units. This applicability and the observations made in Section 6.3 are the main reason why we conducted the following simulation study.

We have seen in Section 6.1 that D and $E(\omega)$ are the mechanisms that result in the highest and nearly the same revenues. Thus, we show simulated revenues arising from *PL* relative to those received by *D*. Similar to the approach used for creating Figure 3, we calculate the percentages of *D*'s revenue *PL* obtains. These percentages are shown in the following figure. We let the ordinate start at 0.5 to demonstrate differences between the curves more clearly.

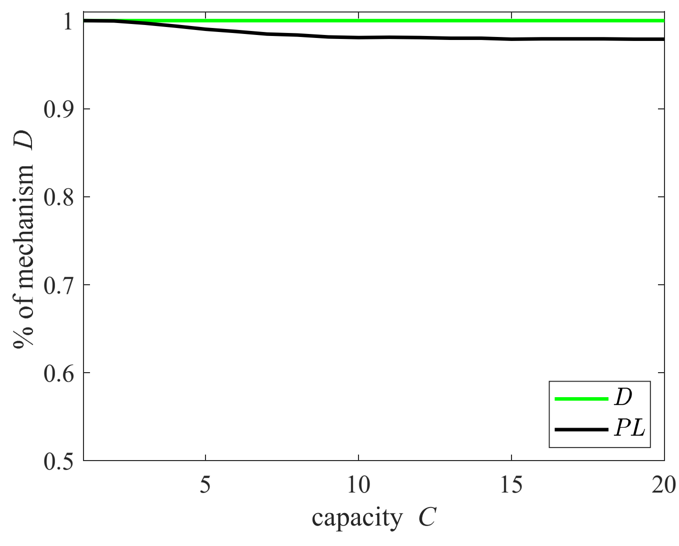

Figure 7: Performance of mechanism PL relative to D for $C \leq 20, T = 10$

Overall, the piecewise-linear pricing scheme performed well. For $C = 1$ and $C = 2$ the results of *D* and *PL* are (as expected) identical. In both cases, there is no actual restrictions on batch prices as there are at most two units that can be sold. For $C \geq 3$ the gap between *D* and *PL* is slowly widening. However, *PL* obtains at least 97.9% of *D*'s revenues.

Apparently, restricting batch prices to a piecewise-linear pricing scheme leads only to a small cutback in revenues. Therefore, it might be favorable in many cases to implement this pricing scheme, as it is easy to communicate and performs very well.

7 Conclusion

In this study, we introduced a nonlinear dynamic pricing model. Our model is based on a customer choice model that captures customers' base willingness-to-pay and consumption indicator. Although the resulting probability function is complex, we were able to simplify it by leveraging structural properties and removing non-essential batch prices from the action space. We proceeded with developing optimality conditions for the stage-wise optimization model and its fluid approximation. However, finding the optimal solution of the stage-wise optimization model remains challenging. To cope with these difficulties, we introduced three novel heuristics: one that uses a decomposition approach and two that calculate expected optimal prices. For every heuristic, we ensured approximate optimality by incorporating the solution of the fluid approximation.

In a simulation study, the heuristics performed very well and quite similar with one of them being only slightly behind the other two. Moreover, they significantly outperformed two approaches that most likely might be applied in practice. We further analyzed both best performing heuristics and found several interesting characteristics of well-performing pricing policies: In cases with reasonably large stocks (in our setting with, e.g., $\frac{C}{T} \geq 2$), batch prices for small batches are slowly decreasing over time. This is particularly interesting as it is an indicator that changing prices at a lower rate (not after every customer) might still perform well in the presence of multiunit demand. This makes the obtained policies also applicable in settings where the firm cannot sustain frequent changes in batch prices due to, e.g., technical reasons, customers' reluctance, or strategic considerations. On the other hand, in scenarios with a limited stock (in our setting with, $\frac{T}{C} \geq 2$), the importance of nonlinear pricing is declining, whereas a typical (standard) dynamic pricing structure becomes more and more relevant.

Another finding is that batch prices are nearly linear for low- and medium-sized batches starting with the two-unit batch. This allows an easy to communicate price structure. Instead of displaying a long list containing prices for every possible batch size, the firm can quote two prices, one for the first unit and one for additional units. Another numerical study verified that this leads only to rather small cutbacks in revenue, offering a pragmatic balance between optimizing returns and operational applicability.

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Supplement: Asymptotically Optimal Solutions for Nonlinear Dynamic Pricing in the Presence of Multiunit Demand

S.1 Proof of Lemma 1

We will show the first part of the lemma in detail. The second part follows by a similar argumentation. For every $i \neq k$ with $i, k > j$ and $r_i - r_j \neq 0 \neq r_k - r_j$, it holds that $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \Leftrightarrow \frac{r_i - r_j}{r_k - r_j} = \frac{\sum_{m=j}^{i-1} l^m}{\sum_{m=j}^{k-1} l^m}$. W.l.o.g., we assume that $i > k$ (otherwise we could focus on $\frac{r_k - r_j}{r_i - r_j} = \frac{\sum_{m=j}^{k-1} l^m}{\sum_{m=j}^{i-1} l^m}$). We define the function $f(l) = \frac{\sum_{m=j}^{i-1} l^m}{\sum_{m=j}^{k-1} l^m}$ and note that $f(l)$ is continuously differentiable. By differentiating $f(l)$, we show that this function is monotonically increasing on the whole interval $(0, 1)$:

Applying the quotient rule, we note that

$$\frac{d}{dl} f(l) > 0 \Leftrightarrow \sum_{m=j}^{k-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{i-1} l^n - \sum_{m=j}^{i-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{k-1} l^n > 0\,.$$

After some rearrangements of the sums, we get

$$\begin{aligned}
&\sum_{m=j}^{k-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{i-1} l^n - \sum_{m=j}^{i-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{k-1} l^n = \\
&\qquad = \sum_{m=1}^{k-j} \sum_{n=1}^{m} (i-n) l^{k+i-2-m} + \sum_{m=k-j+1}^{i-j} \sum_{n=1}^{k-j} (i+k-j-n-m) l^{k+i-2-m} \\
&\qquad + \sum_{m=i-j+1}^{k+i-1-2j} \sum_{n=m}^{k+i-1-2j} (i+k-1-j-n) l^{k+i-2-m} - \sum_{m=1}^{k-j} \sum_{n=1}^{m} (k-n) l^{k+i-2-m} \\
&\qquad - \sum_{m=k-j+1}^{i-j} \sum_{n=1}^{k-j} (k-n) l^{k+i-2-m} - \sum_{m=i-j+1}^{k+i-1-2j} \sum_{n=m}^{k+i-1-2j} (i+k-1-j-n) l^{k+i-2-m} \\
&\qquad = \sum_{m=1}^{k-j} \sum_{n=1}^{m} (i-k) l^{k+i-2-m} + \sum_{m=k-j+1}^{i-j} \sum_{n=1}^{k-j} (i-j-m) l^{k+i-2-m} > 0
\end{aligned}$$

for all $l \in (0, 1)$. Together with the independence of $\frac{r_i - r_j}{r_k - r_j}$ from l, it follows that there is at most one $l \in (0,1)$ such that $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}$.

To show the second part of this lemma, we use a similar argumentation defining $g(l) = \frac{\sum_{m=k}^{j-1} l^m}{\sum_{m=i}^{j-1} l^m} = 1 - \frac{\sum_{m=i}^{k-1} l^m}{\sum_{m=i}^{j-1} l^m}$. □

S.2 Proof of Lemma 2

For any $i > j$ it holds that $\max_{0 \le k \le j-1} \left\{ \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} \right\} \le \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \min_{j+1 \le k \le c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \right\}$ for every $l \in \left[\underline{l}_{ij}^{min}(\boldsymbol{r}), \overline{l}_{ij}^{min}(\boldsymbol{r}) \right]$ and $\max_{0 \le k \le i-1} \left\{ \frac{r_i - r_k}{\sum_{m=k}^{i-1} l^m} \right\} = \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \le \min_{i+1 \le k \le c} \left\{ \frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m}, 1 \right\}$ for every $l \in \left[\underline{l}_{ij}^{max}(\boldsymbol{r}), \overline{l}_{ij}^{max}(\boldsymbol{r}) \right]$. We show the equality of both intervals by two equivalencies. The first one is given by

$$\max_{0 \le k \le j-1} \left\{ \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} \right\} \le \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \Leftrightarrow \max_{0 \le k \le j-1} \left\{ \frac{r_i - r_k}{\sum_{m=k}^{i-1} l^m} \right\} \le \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}$$

and the second one by

$$\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \min_{j+1 \le k \le c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \right\}$$

$$\Leftrightarrow \left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \le \min_{j+1 \le k \le i-1} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \right\} \right) \wedge \left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \le \min_{i+1 \le k \le c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \right\} \right)$$

$$\Leftrightarrow \left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \ge \max_{j+1 \le k \le i-1} \left\{ \frac{r_i - r_k}{\sum_{m=k}^{i-1} l^m} \right\} \right) \wedge \left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \le \min_{i+1 \le k \le c} \left\{ \frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m}, 1 \right\} \right).$$

Both equivalencies together result in the proof of the lemma. To verify both equivalencies, the following corollary might help. □

Corollary 1 *For $j > k > i$, one of the following three cases applies:*

1. $\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} < \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} < \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$
2. $\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} = \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} = \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$
3. $\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} > \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} > \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$

Proof: As $\frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} = \frac{\sum_{m=i}^{k-1} l^m}{\sum_{m=i}^{j-1} l^m} \cdot \frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m} + \frac{\sum_{m=k}^{j-1} l^m}{\sum_{m=i}^{j-1} l^m} \cdot \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$ is a convex combination of $\frac{r_k - r_i}{\sum_{m=i}^{k-1} l^m}$ and $\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}$, the statement above immediately follows. □

S.3 Proof of Lemma 3

To prove the remaining part of Lemma 3, we again assume that our price vector does not follow the given condition, i.e. we choose r_j such that $\left(r_j - r_{j-1}\right)^{\frac{1}{j-1}} > \left(r_{j+1} - r_j\right)^{\frac{1}{j}} \geq 0$. In this case, it holds that $\frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_{j+1} - r_j}{l^j}$ for every $l \in \left[\left(r_{j+1} - r_j\right)^{\frac{1}{j}}, 1\right]$ as:

$$\frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_{j+1} - r_j}{l^j} \Leftrightarrow l > \frac{r_{j+1} - r_j}{r_j - r_{j-1}}$$

With $\frac{r_{j+1} - r_j}{r_j - r_{j-1}} = \left(r_{j+1} - r_j\right)^{\frac{1}{j}} \cdot \left(\frac{\left(r_{j+1} - r_j\right)^{\frac{1}{j}}}{\left(r_j - r_{j-1}\right)^{\frac{1}{j-1}}}\right)^{j-1} < \left(r_{j+1} - r_j\right)^{\frac{1}{j}}$, it holds that the condition of the right side is met for every $l \in \left[\left(r_{j+1} - r_j\right)^{\frac{1}{j}}, 1\right]$. As $\frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_{j+1} - r_j}{l^j}$, it holds that

$$w > \frac{r_j - r_{j-1}}{l^{j-1}} \Rightarrow w > \frac{r_{j+1} - r_j}{l^j}$$

and, thus,

$$w \cdot l^{j-1} > r_j - r_{j-1} \Rightarrow w \cdot l^j > r_{j+1} - r_j.$$

This finally results in $w \cdot \sum_{k=0}^{j} l^k - r_{j+1} \geq w \cdot \sum_{k=0}^{j-1} l^k - r_j$ for every $w \in [0,1], l \in \left[\left(r_{j+1} - r_j\right)^{\frac{1}{j}}, 1\right]$ such that $w \cdot \sum_{k=0}^{j-1} l^k - r_j = \max_{i \leq j}\left\{w \cdot \sum_{k=0}^{i-1} l^k - r_i\right\}$.

With a similar argumentation, we can conclude that $w \cdot \sum_{k=0}^{j-2} l^k - r_{j-1} \geq w \cdot \sum_{k=0}^{j-1} l^k - r_j$ for every $w \in [0,1], l \in \left[0, \left(r_{j+1} - r_j\right)^{\frac{1}{j}}\right]$ such that $w \cdot \sum_{k=0}^{j-1} l^k - r_j = \max_{i \leq j}\left\{w \cdot \sum_{k=0}^{i-1} l^k - r_i\right\}$.

All in all, we can summarize that for every batch size j and a corresponding batch price r_j such that $\left(r_j - r_{j-1}\right)^{\frac{1}{j-1}} > \left(r_{j+1} - r_j\right)^{\frac{1}{j}}$, there is no customer who is willing to purchase j units.

To eliminate demand for j units, the firm could also choose r_j such that $\left(r_j - r_{j-1}\right)^{\frac{1}{j-1}} = \left(r_{j+1} - r_j\right)^{\frac{1}{j}}$. This choice is always possible because of the continuity of prices and $r_{j-1} \leq r_{j+1}$. Thereof, there is no need to considers prices r_j with $\left(r_j - r_{j-1}\right)^{\frac{1}{j-1}} > \left(r_{j+1} - r_j\right)^{\frac{1}{j}}$. □

S.4 Proof of Lemma 4

We recall that $\mathcal{R}_c = \Big\{ \boldsymbol{r} \in \mathbb{R}^c \colon 0 \le r_1 \le \cdots \le r_c \le c,\ r_j \le j\ \ \forall j, \left(r_j - r_{j-1}\right)^{\frac{1}{j-1}} \le 1\ for\ j \ge 2, and\ \left(r_j - r_{j-1}\right)^{\frac{1}{j-1}} \le \left(r_{j+1} - r_j\right)^{\frac{1}{j}}\ for\ \ 2 \le j \le c-1 \Big\}$ and $\Lambda_j(\boldsymbol{r}) = \Big\{ l \in [0,1] \mid \max_{0 \le k \le j-1} \Big\{ \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} \Big\} \le\ 1 = \min_{j+1 \le k \le c} \Big\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \Big\} \Big\} = \big[\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r}) \big]$.

We start with a special case by observing, that for $\boldsymbol{r} \in \mathcal{R}_c$ it holds by definition that $r_{j+1} = r_j \Rightarrow r_j = r_{j-1}$ and $r_j > r_{j-1} \Rightarrow r_{j+1} > r_j$. Thus, either $r_1 < r_2 < \cdots < r_c$ or there is index i with $r_1 = r_2 = \cdots = r_i < r_{i+1} < \cdots < r_c$.

For the letter case, it holds that $\Lambda_1(\boldsymbol{r}) = \Lambda_2(\boldsymbol{r}) = \cdots = \Lambda_{i-1}(\boldsymbol{r}) = \emptyset$. There is no clear choice of $\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r}),\ j < i$. As $\lambda \neq l$ a.s. for any $l \in [0,1]$, we can as well choose $\underline{l}_j(\boldsymbol{r}), = \bar{l}_j(\boldsymbol{r}) = 0$ (for consistency) even though $\big[\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r}) \big] \neq \emptyset$.

In the following, we will concentrate on j with $r_j < r_{j+1}$ regardless of whether j starts at 1 or at the aforementioned i.

Based on $l \in \Lambda_j(\boldsymbol{r}) \Rightarrow 1 = \min_{j+1 \le k \le c} \Big\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \Big\}$, we are interested in the order the lines $f_{kj}(l) = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}$, $k \ge j+1$, are dropping below 1 for given $\boldsymbol{r} \in \mathcal{R}_c$. As $f_{kj}(l)$ is decreasing in $l \in [0,1]$ this happens at most once on $[0,1]$.

First, we note that $\frac{r_{j+1} - r_j}{l^j} = 1 \Leftrightarrow l = \left(r_{j+1} - r_j\right)^{\frac{1}{j}}$ as $r_j < r_{j+1}$. Thereof, we can already conclude from $\boldsymbol{r} \in \mathcal{R}_c$ that $f_{k+1,k}(l)$ drops below 1 before (or at the same time as) $f_{k+2,k+1}(l)$ does, for every $k \ge j$. By Corollary 1, we know that $f_{k+2,k}(l)$ is dropping below 1 between $f_{k+1,k}(l)$ and $f_{k+2,k+1}(l)$. Repeatedly applying Corollary 1 shows that $f_{k+3,k}(l)$ is between $f_{k+2,k}(l)$ and $f_{k+3,k+2}(l)$, $f_{k+4,i}(l)$ is between $f_{k+3,k}(l)$ and $f_{k+4,k+3}(l)$, and so on. We can therefore conclude that the correct order of lines $f_{kj}(l), k > j$, dropping below 1 is $f_{j+1,j}(l), f_{j+2,j}(l), \ldots, f_{cj}(l)$. Most importantly, the last $l \in [0,1]$ with $1 = \min_{j+1 \le k \le c} \{ f_{kj}(l), 1 \}$ is the point where $f_{j+1,j}(l) = 1$. At this point $f_{kj}(l) \ge 1$, $k > j$, because of the given order and the fact that every $f_{kj}(l)$ is decreasing in l. This point is given by $l = \left(r_{j+1} - r_j\right)^{\frac{1}{j}}$ and we know that the condition $1 = \min_{j+1 \le k \le c} \Big\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \Big\}$ is met for every $l \le \left(r_{j+1} - r_j\right)^{\frac{1}{j}}$. We now can set $\bar{l}_j(\boldsymbol{r}) = \left(r_{j+1} - r_j\right)^{\frac{1}{j}}$. For $j = c$, the condition is always met, and we set $\bar{l}_c(\boldsymbol{r}) = 1$.

We now concentrate on $\max_{0 \le k \le j-1} \{f_{jk}(l)\} \le 1$. For $j = 1$, this boils down to $r_1 \le 1$ and is always fulfilled for $\boldsymbol{r} \in \mathcal{R}_c$. In this case we set $\underline{l}_1(\boldsymbol{r}) = 0$. If there is an index i with $r_1 = r_2 = \cdots = r_i < r_{i+1} < \cdots < r_c$, then $\max_{0 \le k \le i-1} \{f_{ik}(l)\} = 0$. In this case, we set $\underline{l}_i(\boldsymbol{r}) = 0$. For the remaining cases, we again apply Corollary 1. With the same argumentation as above, we can conclude that the correct order of lines $f_{jk}(l)$, $j > k$, dropping below 1 is $f_{j0}(l)$, $f_{j1}(l), \dots, f_{j,j-1}(l)$. Some of these $f_{jk}(l)$, $k < j$, might be below 1 for every $l \in [0,1]$. However, this does not affect the proof of this Lemma. Particularly, it holds that $f_{j,j-1}(l) \le 1 \Leftrightarrow f_{jk}(l) \le 1 \; \forall k < j \Leftrightarrow \max_{0 \le k \le j-1} \{f_{jk}(l)\} \le 1$ because of the given order and the fact that every $f_{jk}(l)$ is decreasing in l. Therefore, the condition $\max_{0 \le k \le j-1} \{f_{jk}(l)\} \le 1$ is met for every $l \ge (r_j - r_{j-1})^{\frac{1}{j-1}}$. We now can set $\underline{l}_j(\boldsymbol{r}) = (r_j - r_{j-1})^{\frac{1}{j-1}}$.

To sum up, we have shown that $\underline{l}_1(\boldsymbol{r}) = 0$, $\bar{l}_c(\boldsymbol{r}) = 1$ and $\bar{l}_j(\boldsymbol{r}) = (r_{j+1} - r_j)^{\frac{1}{j}} = \underline{l}_{j+1}(\boldsymbol{r})$ and, thus, the whole lemma. □

S.5 Proof of Lemma 5

Exemplarily, we will walk one of these cases through and assume that $\boldsymbol{i} > \boldsymbol{j}$. The other two would follow in a similar matter and we decided to omit a detailed derivation.

Partially differentiating (5) leads to

$$\frac{d}{d\,r_i}p_j(\boldsymbol{r}) = f_\lambda\left(\underline{l}_{j+1}(\boldsymbol{r})\right)\cdot\frac{d}{d\,r_i}\underline{l}_{j+1}(\boldsymbol{r}) - f_\lambda\left(\underline{l}_j(\boldsymbol{r})\right)\cdot\frac{d}{d\,r_i}\underline{l}_j(\boldsymbol{r})$$

$$+\int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})}\frac{1}{\sum_{m=j}^{i-1} l^m}\, f_\omega\left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}\right) f_\lambda(l)\, dl$$

$$+\sum_{k=j+1}^{c}\left(F_\omega\left(\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m}\right)\cdot f_\lambda\left(\bar{l}_{kj}(\boldsymbol{r})\right)\cdot\frac{d}{d\,r_i}\bar{l}_{kj}(\boldsymbol{r}) - F_\omega\left(\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\underline{l}_{kj}(\boldsymbol{r})\right)^m}\right)\right.$$

$$\left.\cdot f_\lambda\left(\underline{l}_{kj}(\boldsymbol{r})\right)\cdot\frac{d}{d\,r_i}\underline{l}_{kj}(\boldsymbol{r})\right)$$

$$-\sum_{k=0}^{j-1}\left(F_\omega\left(\frac{r_j - r_k}{\sum_{m=k}^{j-1}\left(\bar{l}_{jk}(\boldsymbol{r})\right)^m}\right)\cdot f_\lambda\left(\bar{l}_{jk}(\boldsymbol{r})\right)\cdot\frac{d}{d\,r_i}\bar{l}_{jk}(\boldsymbol{r}) - F_\omega\left(\frac{r_j - r_k}{\sum_{m=k}^{j-1}\left(\underline{l}_{jk}(\boldsymbol{r})\right)^m}\right)\right.$$

$$\left.\cdot f_\lambda\left(\underline{l}_{jk}(\boldsymbol{r})\right)\cdot\frac{d}{d\,r_i}\underline{l}_{jk}(\boldsymbol{r})\right) = \int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})}\frac{1}{\sum_{m=j}^{i-1} l^m}\, f_\omega\left(\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}\right) f_\lambda(l)\, dl$$

The last equation holds because there are many terms that neutralize each other. This can be observed by:

- In Remark 1, we established that $\bar{l}_j^{end}(\boldsymbol{r})$ is the maximum of $\bar{l}_{kj}(\boldsymbol{r})$ for $k > j$ as well as the maximum of $\bar{l}_{jk}(\boldsymbol{r})$ for $k < j$. This can be articulated as $\bar{l}_j^{end}(\boldsymbol{r}) = \max_{k>j}\{\bar{l}_{kj}(\boldsymbol{r})\} = \max_{k<j}\{\bar{l}_{jk}(\boldsymbol{r})\}$. For now, we define $k^{max} = \arg\max_{k>j}\{\bar{l}_{kj}(\boldsymbol{r})\}$ and $k_{max} = \arg\max_{k<j}\{\bar{l}_{jk}(\boldsymbol{r})\}$. Using the same argumentation as outlined in Remark 2, we conclude that $\frac{r_{k^{max}} - r_j}{\sum_{m=j}^{k^{max}-1}\left(\bar{l}_{k^{max}j}(\boldsymbol{r})\right)^m} = \frac{r_j - r_{k_{max}}}{\sum_{m=k_{max}}^{j-1}\left(\bar{l}_{jk_{max}}(\boldsymbol{r})\right)^m}$ and $\frac{d}{d\,r_i}\bar{l}_{k^{max}j}(\boldsymbol{r}) = \frac{d}{d\,r_i}\bar{l}_{jk_{max}}(\boldsymbol{r})$.
- Moreover, we know from Remark 2 that for every $\bar{l}_{kj}(\boldsymbol{r}) \neq \bar{l}_j^{end}(\boldsymbol{r})$ there is $\underline{l}_{hj}(\boldsymbol{r})$, $h > j$, such that $\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m} = \frac{r_h - r_j}{\sum_{m=j}^{h-1}\left(\underline{l}_{hj}(\boldsymbol{r})\right)^m}$ and $\frac{d}{d\,r_i}\bar{l}_{kj}(\boldsymbol{r}) = \frac{d}{d\,r_i}\underline{l}_{hj}(\boldsymbol{r})$.
- The same conclusion holds for $\bar{l}_{jk}(\boldsymbol{r}) \neq \bar{l}_j^{end}(\boldsymbol{r})$ and its corresponding $\underline{l}_{jh}(\boldsymbol{r})$, $h < j$.

- Remark 2 further reveals that $\frac{d}{d\,r_i}\underline{l}_{j+1}(\boldsymbol{r}) = \frac{d}{d\,r_i}\underline{l}_{j+1,j}(\boldsymbol{r})$ and $\frac{r_{j+1}-r_j}{\left(\underline{l}_{j+1,j}(\boldsymbol{r})\right)^j} = 1$. Analogously, it holds that $\frac{d}{d\,r_i}\underline{l}_j(\boldsymbol{r}) = \frac{d}{d\,r_i}\underline{l}_{j,j-1}(\boldsymbol{r})$ and $\frac{r_j-r_{j-1}}{\left(\underline{l}_{j,j-1}(\boldsymbol{r})\right)^{j-1}} = 1$.

Using all of these equations, we get $\frac{d}{d\,r_i}p_j(\boldsymbol{r}) = \int_{\underline{l}_{ij}(r)}^{\bar{l}_{ij}(r)} \frac{1}{\sum_{m=j}^{i-1} l^m} f_\omega\left(\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}\right) f_\lambda(l)\,dl$. With similar steps, we can build the partial differentiations for cases $i < j$ and $i = j$. □

S.6 Proof of Proposition 1

An optimal solution for (11) is either found at the boundary of feasible region $\mathcal{R}_c$ or when it satisfies the first order condition $\frac{d}{d\,r_i}\left(\sum_{k=1}^{c} p_k(\boldsymbol{r}) \cdot \left(r_k - \Delta_k V_{t-1}(c)\right)\right) = 0$ for every i. The condition for optimality at an interior point is straightforward; however, exploring optimality at the boundary requires additional consideration: A solution residing on the boundary is marked by at least one batch size j for which $p_j(\boldsymbol{r}) = 0$. For simplicity, let us initially consider the scenario with only one such batch size j, though the argument can be extended to multiple batch sizes where demand is nullified.

Eliminating the option to purchase j units a priori and reapplying the optimization yields an identical optimality condition, except the summation now omits the index j. This index can be reintegrated into the summation because, by definition $\underline{l}_{jk}(\boldsymbol{r}) = \bar{l}_{jk}(\boldsymbol{r})$ ($\underline{l}_{kj}(\boldsymbol{r}) = \bar{l}_{kj}(\boldsymbol{r})$) for $j > k$ ($j < k$), rendering the integral $\int_{k(\boldsymbol{r})}^{\bar{l}_{jk}(\boldsymbol{r})} \frac{\Delta_j V_{t-1}(c) - \Delta_k V_{t-1}(c)}{\sum_{m=k}^{j-1} l^m} f_\lambda(l)\,dl = 0$ ($\int_{\underline{l}_{kj}(\boldsymbol{r})}^{\bar{l}_{kj}(\boldsymbol{r})} \frac{\Delta_k V_{t-1}(c) - \Delta_j V_{t-1}(c)}{\sum_{m=j}^{k-1} l^m} f_\lambda(l)\,dl = 0$). Furthermore, given the definitions of $\underline{l}_j(\boldsymbol{r})$ and $\underline{l}_{j+1}(\boldsymbol{r})$, it follows that $F_\lambda\left(\underline{l}_{j+1}(\boldsymbol{r})\right) - F_\lambda\left(\underline{l}_j(\boldsymbol{r})\right) = 0$. Combining all these arguments leads to $\left(F_\lambda\left(\underline{l}_{j+1}(\boldsymbol{r})\right) - F_\lambda\left(\underline{l}_j(\boldsymbol{r})\right)\right) - \sum_{k=j+1}^{c} \int_{\underline{l}_{kj}(\boldsymbol{r})}^{\bar{l}_{kj}(\boldsymbol{r})} \frac{\Delta_k V_{t-1}(c) - \Delta_j V_{t-1}(c)}{\sum_{m=j}^{k-1} l^m} f_\lambda(l)\,dl + \sum_{k=0}^{j-1} \int_{\underline{l}_{jk}(\boldsymbol{r})}^{\bar{l}_{jk}(\boldsymbol{r})} \frac{\Delta_j V_{t-1}(c) - \Delta_k V_{t-1}(c)}{\sum_{m=k}^{j-1} l^m} f_\lambda(l)\,dl = 0$. Particularly, it holds for every j with $p_j(\boldsymbol{r}) = 0$ that $\frac{d}{d\,r_j}\left(\sum_{k=1}^{c} p_k(\boldsymbol{r}) \cdot \left(r_k - \Delta_k V_{t-1}(c)\right)\right) = 0$.

In summary, whether at the boundary or an interior point, the optimal solution must satisfy the first order condition $\frac{d}{d\,r_i}\left(\sum_{k=1}^{c} p_k(\boldsymbol{r}) \cdot \left(r_k - \Delta_k V_{t-1}(c)\right)\right) = 0$ for every i, reinforcing the comprehensive nature of the optimality condition across the entire decision space. □

S.7 Proof of Proposition 3

In the following proof, we assume that all three conditions are fulfilled. The proof itself is quite technical, hiding the specific ideas behind a lot of math. In the following lines, we want to show that these conditions lead to a scenario, where only the events of selling $0, j-1$, and $j+1$ units limit the probability of selling j units. We recall that for any given $l \in [0,1]$, selling j units only occur iff

$$\max_{0 \le k \le j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i} \right\} \le w \le \min_{j+1 \le k \le c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}, 1 \right\}.$$

In our first step, we show that conditions 1 and 2 reduces the upper bound $\min_{j+1 \le k \le c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}, 1 \right\}$ to $\min\left\{ \frac{r_{j+1} - r_j}{l^j}, 1 \right\}$. Thereby, we prove that in the customer choice model the options of purchasing more than $j+1$ units are in no direct competition with the option of purchasing j units. Analogously, we determine that the lower bound reduces to $\max\left\{ \frac{r_j - r_{j-1}}{l^{j-1}}, \frac{r_j}{\sum_{m=0}^{j-1} l^m} \right\}$, leaving only the events of selling 0 or $j-1$ units as relevant options for customers, who may purchase j units.

1. From condition 1, we know that for any j it holds that $\frac{r_{j+1} - r_j}{l^j} \ge \frac{r_j - r_{j-1}}{l^{j-1}}$ for every $l \le \bar{l}_j^{end}(\boldsymbol{r})$ Additionally, in combination with condition 2, we also know that $\frac{r_{i+1} - r_i}{l^i} \ge \frac{r_i - r_{i-1}}{l^{i-1}}$ for every $i \ge j$ and $l \le \bar{l}_j^{end}(\boldsymbol{r})$. Consequently, it also follows that $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} \ge \frac{r_{j+1} - r_j}{l^j}$ for every $l \le \bar{l}_j^{end}(\boldsymbol{r})$. Which ultimately yields $\min_{j+1 \le i \le c} \left\{ \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}, 1 \right\} = \min\left\{ \frac{r_{j+1} - r_j}{l^j}, 1 \right\}$ for every $l \le \bar{l}_j^{end}(\boldsymbol{r})$.
2. Condition 3 implies that $\bar{l}_j^{end}(\boldsymbol{r})$ is defined by the two specific lines, $\frac{r_{j+1} - r_j}{l^j}$ (the remaining part of the upper bound) and $\frac{r_j}{\sum_{m=0}^{j-1} l^m}$ (part of the lower bound). Thereby, we know that $\frac{r_j}{\sum_{m=0}^{j-1} l^m} = \max_{0 \le i \le j-1} \left\{ \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} \right\}$ if $l = \bar{l}_j^{end}(\boldsymbol{r})$.

   To prove $\max_{0 \le i \le j} \left\{ \frac{r_{j+1} - r_i}{\sum_{m=i}^{j} l^m} \right\} = \max\left\{ \frac{r_{j+1} - r_j}{l^j}, \frac{r_{j+1}}{\sum_{m=0}^{j} l^m} \right\}$ for $l \le \bar{l}_{j+1}^{end}(\boldsymbol{r})$, we split the interval $\left[0, \bar{l}_{j+1}^{end}(\boldsymbol{r})\right]$ in three parts: $\left[0, \bar{l}_j^{end}(\boldsymbol{r})\right)$, $\bar{l}_j^{end}(\boldsymbol{r})$, and $\left(\bar{l}_j^{end}(\boldsymbol{r}), \bar{l}_{j+1}^{end}(\boldsymbol{r})\right]$ and show that

$\max_{0\le i\le j}\left\{\frac{r_{j+1}-r_i}{\sum_{m=i}^{j} l^m}\right\} = \max\left\{\frac{r_{j+1}-r_j}{l^j}, \frac{r_{j+1}}{\sum_{m=0}^{j} l^m}\right\}$ holds on all three parts. We know from the definition of $\bar{l}_j^{end}(\boldsymbol{r})$ that $\frac{r_{j+1}-r_j}{l^j} > \max_{0\le i\le j-1}\left\{\frac{r_j-r_i}{\sum_{m=i}^{j-1} l^m}\right\}$ for $l < \bar{l}_j^{end}(\boldsymbol{r})$. With some calculus, this order carries over to $\frac{r_{j+1}-r_j}{l^j} > \max_{0\le i\le j-1}\left\{\frac{r_{j+1}-r_i}{\sum_{m=i}^{j+1} l^m}\right\}$ for $l < \bar{l}_j^{end}(\boldsymbol{r})$. Consequently, it holds that $\frac{r_{j+1}-r_j}{l^j} \ge \max_{0\le i\le j}\left\{\frac{r_{j+1}-r_i}{\sum_{m=i}^{j+1} l^m}\right\}$ for $l < \bar{l}_j^{end}(\boldsymbol{r})$, proving the first part. For the second part, $l = \bar{l}_j^{end}(\boldsymbol{r})$, we know that $\frac{r_{j+1}-r_j}{l^j} = \frac{r_j}{\sum_{m=0}^{j-1} l^m}$, which can be extended to $\frac{r_{j+1}}{\sum_{m=0}^{j} l^m} = \frac{r_j}{\sum_{m=0}^{j-1} l^m} = \frac{r_{j+1}-r_j}{l^j}$. With continuity of the lines, it also holds that $\frac{r_{j+1}}{\sum_{m=0}^{j} l^m} = \frac{r_{j+1}-r_j}{l^j} \ge \max_{0\le i\le j}\left\{\frac{r_{j+1}-r_i}{\sum_{m=i}^{j+1} l^m}\right\}$ for $l = \bar{l}_j^{end}(\boldsymbol{r})$. For the third and last part, we make use of Lemma 1, which outlines that lines could only cross each other at most once. As $\frac{r_{j+1}-r_j}{l^j} = \max_{0\le i\le j}\left\{\frac{r_{j+1}-r_i}{\sum_{m=i}^{j+1} l^m}\right\}$ for $l = \bar{l}_j^{end}(\boldsymbol{r})$ and for $l = \bar{l}_{j+1}^{end}(\boldsymbol{r})$, as per condition 3, it follows that $\frac{r_{j+1}-r_j}{l^j} = \max_{0\le i\le j}\left\{\frac{r_{j+1}-r_i}{\sum_{m=i}^{j+1} l^m}\right\}$ for every $\bar{l}_j^{end}(\boldsymbol{r}) \le l \le \bar{l}_{j+1}^{end}(\boldsymbol{r})$. Otherwise, there would be two lines which intersect at least two times.

With these two steps, we have shown that batch prices that fulfill the stated conditions lead to a certain structure of the probability function. Thereby, the probability of selling j units is only limited by three events: selling 0, $j-1$, and $j+1$ units. □

S.8 Calculation of $\boldsymbol{p_j^{d,a}(r)}$ for $\boldsymbol{\omega, \lambda \sim U[0,1]}$

We now only have two possibilities for $\bar{l}_j^{end}(\boldsymbol{r})$ (cf. Remark 1):

1. $\frac{r_{j+1}-r_j}{r_j-r_{j-1}}$

   or

2. l such that $\frac{r_{j+1}-r_j}{l^j} = \frac{r_j}{\sum_{m=0}^{j-1} l^m}$

The lower of these two possible values is $\bar{l}_j^{end}(\boldsymbol{r})$. If the first case applies, it holds that $\Lambda_{j0}^{max}(\boldsymbol{r}) = \emptyset$, $\Lambda_{j,j-1}^{max}(\boldsymbol{r}) = \left[\underline{l}_j(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$, and $\Lambda_{j+1,j}^{min}(\boldsymbol{r}) = \left[\underline{l}_{j+1}(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$. If the second case applies, it holds that $\Lambda_{j0}^{max}(\boldsymbol{r}) = \left[\bar{l}_{j-1}^{\mathrm{end}}(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$, $\Lambda_{j,j-1}^{max}(\boldsymbol{r}) = \left[\underline{l}_j(\boldsymbol{r}), \bar{l}_{j-1}^{end}(\boldsymbol{r})\right]$, and $\Lambda_{j+1,j}^{min}(\boldsymbol{r}) = \left[\underline{l}_{j+1}(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$. To express the demand function for both cases with one formulation, we bring back the notation $\underline{l}_{ji}^{max}(\boldsymbol{r})$ and $\bar{l}_{ji}^{max}(\boldsymbol{r})$. In the first case, we choose $\underline{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_j(\boldsymbol{r})$ and $\bar{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_{j0}^{max}(\boldsymbol{r}) = \bar{l}_{j0}^{max}(\boldsymbol{r}) = \bar{l}_j^{end}(\boldsymbol{r})$. In the second case, we choose $\underline{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_j(\boldsymbol{r})$, $\bar{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_{j0}^{max}(\boldsymbol{r}) = \bar{\mathrm{l}}_{j-1}^{\mathrm{end}}(\boldsymbol{r})$, and $\bar{l}_{j0}^{max}(\boldsymbol{r}) = \bar{l}_j^{end}(\boldsymbol{r})$.

Now, we can write $p_1^{d,a}(\boldsymbol{r}) = -(r_2 - r_1) \cdot \log(r_1)$, $p_2^{d,a}(\boldsymbol{r}) = -(r_3 - r_2) \cdot \left(\frac{1}{\bar{l}_2^{\mathrm{end}}(\boldsymbol{r})} - \frac{1}{\underline{l}_3(\boldsymbol{r})}\right) + \underline{l}_3(\boldsymbol{r}) - \underline{l}_2(\boldsymbol{r}) - r_2 \cdot \left(\log\left(1 + \bar{l}_2^{end}(\boldsymbol{r})\right) - \log\left(1 + \underline{l}_{20}^{max}(\boldsymbol{r})\right)\right) + (r_2 - r_1) \cdot \log(r_1)$, and $p_j^{d,a}(\boldsymbol{r}) = -\frac{r_{j+1} - r_j}{j-1} \cdot \left(\frac{1}{\left(\bar{l}_j^{\mathrm{end}}(\boldsymbol{r})\right)^{j-1}} - \frac{1}{\left(\underline{l}_{j+1}(\boldsymbol{r})\right)^{j-1}}\right) + \underline{l}_{j+1}(\boldsymbol{r}) - \underline{l}_j(\boldsymbol{r}) - \int_{\underline{l}_{j0}^{max}(\boldsymbol{r})}^{\bar{l}_j^{\mathrm{end}}(\boldsymbol{r})} \frac{r_j}{\sum_{m=0}^{j-1} l^m} dl + \frac{r_j - r_{j-1}}{j-2} \cdot \left(\frac{1}{\left(\bar{l}_{j,j-1}^{max}(\boldsymbol{r})\right)^{j-2}} - \frac{1}{\left(\underline{l}_j(\boldsymbol{r})\right)^{j-2}}\right)$.