Approximately Optimal Solutions for Nonlinear Dynamic Pricing

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Abstract

We consider a dynamic pricing setting where a firm sells a perishable product over a finite selling horizon. Different from the standard setting, the firm faces multiunit demand and can separately quote a price for every batch size. Customers differ in their attraction to the product and their preference regarding the batch size. These two attributes are depicted by a random variable each and are the basis for the calculation of customers' willingness-to-pay. The resulting customer choice model is very challenging to work with and we put some effort into reducing its complexity.

We develop optimality conditions for the stage-wise optimization problem. As finding the optimal solution in every state is non-trivial, we resort to formulating a fluid approximation model. With a simplifying assumption, we can solve this approximation and subsequently verify that this assumption indeed holds for the optimal solution. The resulting static pricing policy is approximately optimal in our dynamic setting. However, instead of applying this static policy, we use it to ensure approximate optimality of the three novel heuristics we developed in this paper.

We test all heuristics in a simulation study against an upper bound and analyze patterns in the corresponding policies to gain managerial insights. For example, we find that a piecewise linear pricing structure performs very well and might be an easy-to-communicate alternative to full nonlinearity.

Keywords: Revenue Management, Dynamic Pricing, Nonlinear Pricing, Multiunit Purchases, Customer Choice

1 Introduction

Nonlinear pricing has been a common practice in retail and other industries for some time, offering customers special deals like "buy 3, pay 2" or volume discounts. In recent years, the possibility of dynamic price adjustments - enabled by e-commerce and digital price tags - has led to increased interest in this field. However, standard dynamic pricing assumes customers buy only one unit at a time, which is suboptimal for products where batch size is an important factor, such as groceries and clothing.

In this paper, we combine both pricing schemes to nonlinear dynamic pricing. Thereby, we consider the most general case of nonlinear pricing by dynamically quoting a price for every quantity (batch size) of a single product to maximize revenue. As usual in dynamic pricing, the selling horizon and product inventory are limited. Customers decide upon the quantity to purchase after observing prices. Based on customers' (stochastic) decision, the firm immediately earns the corresponding batch price, and the inventory is reduced by batch size. The remaining inventory can be offered to the next customer with new batch prices. Future customers lead to stochastic future earnings that are incorporated into the stage-wise optimization as expected revenue-to-go from remaining inventory and time.

We assume customers strive to maximize their utility which is the difference between willingness-to-pay and price. In our approach, we model customers' willingness-to-pay as a personal, unknown variable that depends on two customer attributes: their attraction to and consumption of the product. These attributes are modelled by two random variables: the base willingness-to-pay and the consumption indicator.

Our contribution adds to the sparse literature on multiunit dynamic pricing, offering insights into the more complex optimization problem that arises when the firm can dynamically quote batch prices. We develop three asymptotically optimal heuristics and discuss pricing patterns of these heuristics in a numerical study.

1.1 Results

Due to the complex customer choice model, finding a closed-form expression of the optimal solution is intractable and beyond the scope of this paper. However, we prove some structural properties to reduce complexity. We then develop optimality conditions for the stage-wise optimization problem. Finding the optimal solution in every stage is challenging, so we introduce three algorithms that heuristically solve the optimization model. Additionally, we ensure that the solution of these algorithms is approximately optimal.

Two of these algorithms make use of the results of Schur (2023) where it is assumed that the firm can observe some of the customers' private information. They developed a (implicit) formulation of optimal prices depending on the realization of base willingness-to-pay and consumption indicator, respectively. We use these dependent optimal batch prices to derive "expected optimal batch prices" to get an approximately optimal solution for our case without observable information.

The third algorithm uses an approach based on a decomposition in units. Instead of modelling customer choice as picking one of several batch sizes, the algorithm assumes that customers can decide for every unit separately whether they want to buy it or not. Thereby, the resulting optimization model greatly simplifies in comparison to the original model and can be solved (at least numerically). The optimal solution to this new optimization model can then be a proxy for our original optimization problem.

Finally, we conduct a simulation study to test our algorithms against an upper bound and analyse patterns in the corresponding policies to get managerial insights.

1.2 Contribution and outline

These results have an impact in several ways. By showing structural properties, we lay the groundwork for applying a realistic customer choice model in a nonlinear dynamic pricing setting. This enables us to develop conditions an optimal solution must meet. Even though finding an optimal solution in every stage is out of reach, we can use the optimality conditions to solve a fluid approximation. Using this solution, we can ensure approximate optimality of our heuristics.

All presented heuristics are performing well in a simulation study and provide batch prices that can be implemented by the firm. Moreover, the conducted simulation study provides further managerial insights. For example, we can observe that a piecewise linear pricing structure leads to only small losses in revenue. With this, a firm can decide if this easy-to-communicate pricing structure is worth the corresponding loss in revenue.

This paper is organized as follows: After giving a short overview of relevant literature, we introduce the customer choice model as well as the optimization problem in Section 3. In the same section, we show some properties of the choice model. In Section 4, we reduce the action space to lower the complexity of the choice model. We proceed with showing optimality conditions regarding the optimization problem and building a fluid model whose solution is approximately optimal for our problem. In Section 5, we introduce three novel heuristics to solve our multiunit dynamic pricing problem approximately optimal. These heuristics are finally tested and analysed in a simulation study.

2 Literature review

Nonlinear pricing can be observed in many industries including, e.g., telecommunications, transportation, energy, supply chains, and retail. Consequently, there is a diverse body of literature on this pricing scheme. Wilson (1993) gives an overview of fields of application, substantial economics, and marketing literature. Although most of this literature considers static pricing, some researchers focus on a dynamic environment and, thus, are more relevant to our case (e.g., Dhebar and Oren 1986, and Braden and Oren 1994).

The roots of dynamic pricing can be traced back to the study of intertemporal price discrimination (see, e.g., Stokey 1979, Landsberger and Meilijson 1985, and Wilson 1988) 30 to 40 years ago. Gallego and van Ryzin (1994) were the first to consider optimal dynamic pricing of a single product with stochastic demand over a finite selling horizon. Since then, a tremendous amount of follow-up papers was published. Several authors structured and summarized this research in review articles (e.g., Bitran and Caldentey 2003, Chiang et al. 2007, and, with a special focus, Gönsch et al. 2013 and den Boer 2015) as well as textbooks (e.g. Talluri and van Ryzin 2004 (Chapter 5) and Phillips 2005 (Chapter 10)).

Whereas there is a large body of literature on dynamic pricing, the consideration of multiunit dynamic pricing constitutes a rather new field. Elmaghraby et al. (2008) analyze the optimal design of a markdown pricing mechanism in the presence of multiunit demand. They consider a complete information setting in the sense that the firm knows all customers and their willingness-to-pay beforehand. Levin et al. (2014) consider a dynamic pricing model where customers' demand is stochastic and occurs in batches. More precisely, customers request batches with a specific number of items. These requests are modeled via a statistically independent counting process. Observing a request, the seller quotes a specific batch price, and then, the customer either purchases this batch with a price-dependent probability or not. Different from their case, in our paper, customers are not predetermined to a specific batch but choose the amount that maximizes their utility based on the quoted prices. Gallego et al. (2020) investigate three dynamic pricing strategies: nonlinear, linear, and block pricing. The choice model is based on customers' aim at utility maximization and customers are characterized by a single random variable. The authors develop optimality conditions and show structural properties. Schur (2023) considers nearly the same setting as we do. The difference is that in their work, the firm can observe some (or all) private information of the next customer in line. Thereby, they consider different forms of personalized pricing and investigate the value of information.

Multiunit dynamic pricing can be compared to the more explored field of multiproduct dynamic pricing by defining batches of a single product as several different "products". Several articles (see, e.g., Zhang and Cooper 2009, Dong et al. 2009, and Akçay et al. 2010, or, for a review, Chen and Chen 2015) consider dynamic pricing of substitutes. In these publications, a customer can choose between several products and each of these products has its resources. Although multiunit dynamic pricing is related to multiproduct dynamic pricing of substitutes, the inventory structure often differs. Several analytical difficulties arise in a setting where "products" consume a different amount of a single resource compared to a setting where each product has its independent resources. Maglaras and Meissner (2006) consider a slightly different multiproduct model where each product consumes one unit of a single resource. They show that, in their setting, dynamic pricing and capacity control can be reduced to a common formulation in which the firm controls the consumption rate of every product regarding resource capacity. Moreover, they prove that the solution of a fluid model is approximately optimal. This proof is generalized to a setting where products can consume more than just one unit of a single resource.

3 Problem definition

In Section 3.1, we introduce the general setting and notation. In Section 3.2, we present the customer choice model where we explore its structural properties and use them to create a functional formulation of the selling probabilities. Lastly, in Section 3.3, we present the optimization model.

3.1 General setting and notation

We adapt the standard setting of dynamic pricing to cope with multiunit purchases. To do so, we consider the following framework: A firm sells a single product over a finite selling horizon. The selling horizon is divided into T periods and indexed backward in time, i.e., periods T and 0 mark the beginning and the end of the horizon, respectively. The initial stock of C units of the product is nonreplenishable and any capacity remaining after the selling horizon is worthless. We assume that exactly one customer arrives in each period $t \in \{T, \dots, 1\}$. At this moment, the firm knows the remaining capacity $c \in \{1, \dots, C\}$ and sets a price vector $\boldsymbol{r} = (r_1, r_2, \dots, r_c)^T$ with r_j marking the price the customer has to pay for j units of the product, i.e. for a batch with batch size j. Depending on the prices the firm quotes, this customer may purchase zero, one or more (up to c) units of the product. Thereby, $p_j(\boldsymbol{r})$ denotes the probability that this customer chooses to buy j units of the product.

3.2 Customer Choice Model

We model customer choice based on the standard random utility model. Thereby, we define utility as the difference between willingness-to-pay (WTP) and price. Precisely, the random utility for purchasing j units for a price vector $\boldsymbol{r}$, i.e. $u_j(\boldsymbol{r})$, is given by

$$u_j(\boldsymbol{r}) = X_j - r_j \qquad \text{for } j = 1, \dots, c, \qquad (1)$$

where X_j is a random variable denoting the willingness-to-pay for j units. We assume that customers rationally maximize their utility. Thus, a customer purchases j units if and only if $u_j(\boldsymbol{r}) = \max_{j=0,\dots,c} \{u_j(\boldsymbol{r})\}$ with $u_0(\boldsymbol{r}) = 0$ denoting the no-purchase option. Intuitively, the willingness-to-pay should be increasing in the number of units. The following model is based on a formulation of Iyengar and Jedidi (2012):

$$X_j = \omega \cdot \sum_{k=0}^{j-1} (\lambda)^k \qquad \text{for } j = 1, \dots, c, \qquad (2)$$

with independent random variables ω and λ.

The first random variable, $\omega \geq 0$, equals the willingness-to-pay for the first unit of the product and influences the willingness-to-pay for bigger batches in a linear manner. We call this parameter base willingness-to-pay and can interpret it as the attractiveness of the product to the customer. The second variable, the consumption indicator λ, has no influence on the willingness-to-pay for the first unit, but controls the rate at which the marginal willingness-to-pay for an additional unit declines. High (low) values of λ imply customers' willingness to stockpile or consume is at a high (low) rate.

To ensure an intuitive structure, we assume ω and λ follow a uniform distribution with support $[0, 1]$, respectively. Thus, X_j is increasing in j with decreasing marginal values (a.s.). The choice probability for a customer to purchase j units is now given by

$$p_j(\boldsymbol{r}) = \int_0^1 \int_0^1 1_{\left\{u_j(\boldsymbol{r}) = \max_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}\right\}}(w, l)\, dw\, dl \qquad \text{for } j = 1, \dots, c, \tag{3}$$

with the indicator function $1_{\left\{u_j(\boldsymbol{r}) = \max_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}\right\}}(w, l)$ that equals 1 if $u_j(\boldsymbol{r}) = \max_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}$. In other words, the indicator function equals 1 if $w \cdot \sum_{k=0}^{j-1} l^k - r_j \geq w \cdot \sum_{k=0}^{i-1} l^k - r_i$ for $i = 1, \dots, c$ and $w \cdot \sum_{k=0}^{j-1} l^k - r_j \geq 0$. Otherwise, the indicator function takes value 0.

Another way to express these selection criteria is: A customer with attributes $w, l \in [0,1]$ purchases j units iff $\max_{0 \leq k \leq j-1}\left\{\frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i}\right\} \leq w \leq \min_{j+1 \leq k \leq c}\left\{\frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}\right\}$.

To keep notation as short as possible, we establish the following conventions: $r_0 = 0$, $\frac{r_1 - r_0}{\sum_{i=0}^{0} l^i} = r_1$ for $l = 0$, $\max_{0 \leq k \leq j-1}\left\{\frac{r_j - r_k}{\sum_{i=k}^{j-1} 0^i}\right\} = \infty$ for $j \neq 1$, and $\min_{j+1 \leq k \leq c}\left\{\frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}, 1\right\} = 1$ for $j = c$.

By utilizing this formulation, we can simplify the choice probability for selling j units to

$$p_j(\boldsymbol{r}) = \int_0^1 \left(\min_{j+1 \leq k \leq c}\left\{\frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}, 1\right\} - \max_{0 \leq k \leq j-1}\left\{\frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i}\right\} \right)^+ dl \qquad \text{for } j = 1, \dots, c, \tag{4}$$

with $(\cdot)^+ = \max\{\cdot, 0\}$. Even though we got rid of one of the integrals, this probability function is still too complex to work with. In particular, it is inconvenient to drag the minimum and maximum functions along. Therefore, we want to split the region we are integrating over in such a way that we can replace $\min_{j+1 \leq k \leq c}\left\{\frac{r_k - r_j}{\sum_{i=j}^{k-1} l^i}, 1\right\}$ and $\max_{0 \leq k \leq j-1}\left\{\frac{r_j - r_k}{\sum_{i=k}^{j-1} l^i}\right\}$.

We start with introducing sets

$$\Lambda_j(\boldsymbol{r}) = \left\{ l \in [0, 1] \mid \max_{0 \leq k \leq j-1}\left\{\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}\right\} \leq 1 = \min_{j+1 \leq k \leq c}\left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1\right\} \right\},$$

$$\Lambda_{ij}^{min}(\boldsymbol{r}) = \left\{ l \in [0, 1] \mid \max_{0 \leq k \leq j-1}\left\{\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}\right\} \leq \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \min_{j+1 \leq k \leq c}\left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1\right\} \right\}, \; i > j, \text{ and}$$

$$\Lambda_{ji}^{max}(\boldsymbol{r}) = \left\{ l \in [0, 1] \mid \max_{0 \leq k \leq j-1}\left\{\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m}\right\} = \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m} \leq \min_{j+1 \leq k \leq c}\left\{\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1\right\} \right\}, \; i < j.$$

These sets are either empty or an interval (not a union of intervals). This follows immediately from the following lemma.

Lemma 1 *For every $i \neq k$ with $i,k > j$ and $r_i - r_j \neq 0 \neq r_k - r_j$, there is at most one $l \in (0,1)$ where $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}$. For every $i \neq k$ with $i,k < j$ and $r_i - r_j \neq 0 \neq r_k - r_j$, there is at most one $l \in [0,1]$ where $\frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} = \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m}$.*

Proof: See Supplement S.1.

Lemma 1, combined with the fact that $\frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} = 1$ holds for at most one $l \in [0,1]$, implies that each curve can only be the minimum or maximum once. At some point, it will intersect with another curve and will always remain above or below this curve. For every set $\Lambda_j(\boldsymbol{r})$, $\Lambda_{ij}^{min}(\boldsymbol{r})$ ($i > j$), and $\Lambda_{ji}^{max}(\boldsymbol{r})$ ($j > i$), we write $[\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r})]$, $[\underline{l}_{ij}^{min}(\boldsymbol{r}), \bar{l}_{ij}^{min}(\boldsymbol{r})]$, and $[\underline{l}_{ji}^{max}(\boldsymbol{r}), \bar{l}_{ji}^{max}(\boldsymbol{r})]$, respectively. We keep in mind that some of these sets might be empty and that the boundaries must be picked accordingly.

Remark 1 *By definition, there is $\bar{l}_j^{end}(\boldsymbol{r})$ such that $\bar{l}_j^{end}(\boldsymbol{r}) = \max_{i>j}\{\bar{l}_{ij}^{min}(\boldsymbol{r})\} = \max_{i<j}\{\bar{l}_{ji}^{max}(\boldsymbol{r})\}$. Moreover, it holds that:*

1. $\bigcup_{i>j} [\underline{l}_{ij}^{min}(\boldsymbol{r}), \bar{l}_{ij}^{min}(\boldsymbol{r})] = [\bar{l}_j(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})]$
2. $\bigcup_{i<j} [\underline{l}_{ji}^{max}(\boldsymbol{r}), \bar{l}_{ji}^{max}(\boldsymbol{r})] = [\underline{l}_j(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})]$

At the moment, we need a lot of notation to express the probability function adequately. However, the following lemma will help us reduce the number of needed notation and will increase conciseness.

Lemma 2 *For every $i > j$ it holds that $[\underline{l}_{ij}^{min}(\boldsymbol{r}), \bar{l}_{ij}^{min}(\boldsymbol{r})] = [\underline{l}_{ij}^{max}(\boldsymbol{r}), \bar{l}_{ij}^{max}(\boldsymbol{r})]$.*

Proof: See Supplement S.2.

We can now shorten our notation to $\underline{l}_{ij}(\boldsymbol{r}) = \underline{l}_{ij}^{min}(\boldsymbol{r}) = \underline{l}_{ij}^{max}(\boldsymbol{r})$ and $\bar{l}_{ij}(\boldsymbol{r}) = \bar{l}_{ij}^{min}(\boldsymbol{r}) = \bar{l}_{ij}^{max}(\boldsymbol{r})$. The probability function can be written as:

$$p_j(\boldsymbol{r}) = \int_{\underline{l}_j(r)}^{\bar{l}_j(r)} 1\, dl + \sum_{i=j+1}^{c} \int_{\underline{l}_{ij}(r)}^{\bar{l}_{ij}(r)} \frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}\, dl - \sum_{i=0}^{j-1} \int_{\underline{l}_{ji}(r)}^{\bar{l}_{ji}(r)} \frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m}\, dl \text{ for } j = 1, \dots, c. \quad (5)$$

3.3 Dynamic programming formulation

In traditional singleunit dynamic pricing, a firm maximizes total expected revenue over the entire selling horizon:

$$V_T^{SU}(C) = \max_{r_{1t} \geq 0\ \forall t \in \{1,\dots,T\}} \left\{\mathbb{E}\left[\sum_{t=1}^{T} r_{1t} \cdot 1_{\{w \geq r_{1t}\}}(w)\right] : \sum_{t=1}^{T} 1_{\{w \geq r_{1t}\}}(w) \leq C \text{ a.s.}\right\}. \quad (6)$$

This optimization problem can be reformulated as the following dynamic program (see, e.g., Talluri and van Ryzin 2004):

$$V_t^{SU}(c) = \max_{r_1 \geq 0} \left\{p_1(r_1) \cdot \left(r_1 + V_{t-1}^{SU}(c-1)\right) + \left(1 - p_1(r_1)\right) \cdot V_{t-1}^{SU}(c)\right\}, \quad (7)$$

with boundary conditions $V_0^{SU}(c) = 0$ for $c \geq 0$ and $V_t^{SU}(0) = 0$ for $t \geq 0$. Here, $V_t^{SU}(c)$ denotes the optimal expected revenue-to-go from period t onwards. The expectation captures two possible events: A sale of one unit occurs with probability $p_1(r_1)$ and the firm immediately obtains a revenue of r_1 and additionally expects a revenue of $V_{t-1}^{SU}(c-1)$ with a reduced stock of $c-1$ units from the next period onwards. No sale occurs with probability $1 - p_1(r_1)$. In this case, the firm expects a revenue of $V_{t-1}^{SU}(c)$ from stock c.

Building on the formulations (6) and (7), we can develop the corresponding multiunit dynamic pricing formulation. As the remaining capacity c (and, thus, the maximum number of purchasable units) varies over the time, we define a state-dependent action space $\mathcal{R}_c = \{\boldsymbol{r} \in \mathbb{R}^c : r_j \geq 0, j = 1, \dots, c\}$ with $\mathcal{R}_0 = \emptyset$. The overall target is still the maximization of the total expected revenue:

$$V_T(C) = \max_{r_t \in \mathcal{R}_C \, \forall t \in \{1,\dots,T\}} \left\{ \mathbb{E}\left[\sum_{t=1}^{T} \sum_{j=1}^{C} r_{jt} \cdot 1_{\left\{u_{jt}(r_t) = \max\limits_{j=0,\dots,c} \{u_{jt}(r_t)\}\right\}}(w,l) \right] : \sum_{t=1}^{T} \sum_{j=1}^{C} j \cdot 1_{\left\{u_{jt}(r_t) = \max\limits_{j=0,\dots,c} \{u_{jt}(r_t)\}\right\}}(w,l) \leq C \text{ a.s.} \right\} \tag{8}$$

and can be achieved by using the following dynamic program:

$$V_t(c) = \max_{\boldsymbol{r} \in \mathcal{R}_c} \left\{ \sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j + V_{t-1}(c-j)\right) + \left(1 - \sum_{j=1}^{c} p_j(\boldsymbol{r})\right) \cdot V_{t-1}(c) \right\}, \tag{9}$$

with boundary conditions $V_0(c) = 0$ for $c \geq 0$ and $V_t(0) = 0$ for $t \geq 0$. Again, $V_t(c)$ denotes the optimal expected revenue-to-go from period t onwards (before the arrival of a customer in t).

With the new formulation, we consider that customers might be willing to buy more than just one unit. Moreover, we allow the firm to set batch prices to take full advantage of a nonlinear pricing scheme. The expectation now captures $c+1$ possible events: With probability $p_j(\boldsymbol{r})$, the firm sells j units and immediately earns r_j. Additionally, it can expect future revenues amounting to $V_{t-1}(c-j)$ with a reduced stock of $c-j$ units from the next period onwards. With probability $1 - \sum_{j=1}^{c} p_j(\boldsymbol{r})$, the firm sells nothing and only faces expected revenue $V_{t-1}(c)$ from stock c and period $t-1$ onwards. We denote the optimal batch prices selected in a state (t,c) by $\boldsymbol{r}_t(c) \in \mathcal{R}_c$.

After defining the difference functions of $V_t(c)$ with respect to c and j, i.e.

$$\Delta_j V_t(c) = V_t(c) - V_t(c-j) \qquad \text{for } j = 1, \dots, c, \tag{10}$$

we can rewrite the optimality equation (9) as

$$V_t(c) = \max_{\boldsymbol{r} \in \mathcal{R}_c} \left\{ \sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right) \right\} + V_{t-1}(c). \tag{11}$$

As can be seen, the optimization breaks down to maximizing the expected additional gain realized by selling something between 1 and c units in period t instead of retaining the capacity for later sales. We refer to $\Delta_j V_{t-1}(c)$ as the opportunity costs for selling j units.

4 Optimality conditions, fluid approximation, and approximate optimality

In this section, we will reduce the action space to exclude irrelevant prices. Thereby, we can refine the definition of the selling probabilities. With this refinement, we can develop optimality conditions to solve the optimization problem. Thereafter, we present a fluid approximation and elaborate on approximate optimality of its solution.

4.1 Action space reduction

Based on the choice model, there is only a specific subset of prices to consider. In the following, we will update the definition of $\mathcal{R}_c$ to exclude irrelevant prices.

The argumentation about irrelevance of certain prices is as follows: Having multiple prices r_j to eliminate demand for j units (i.e. $p_j(\boldsymbol{r}) = 0$) is not necessary. There is only need for one such r_j (depending on $r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_c$) to keep the possibility of pricing out j units available.

First, we can exclude any price r_j with $r_j > r_{j+1}$: Purchasing $j+1$ units almost surely results in higher utility for customers ($\omega \cdot \lambda^j \geq 0$). Consequently, customers would not pay a higher price for j units than for $j+1$ units. Thus, for $r_j > r_{j+1}$, it holds that $p_j(\boldsymbol{r}) = 0$. The same effect can be achieved by setting $r_j = r_{j+1}$ making prices r_j with $r_j > r_{j+1}$ irrelevant.

Second, we can exclude any price r_j with $r_j > j$. Based on $\omega \sum_{k=0}^{j-1} \lambda^k \leq j \leq r_j \Rightarrow p_j(\boldsymbol{r}) = 0$, we can drop $r_j > j$ and still have the possibility to get $p_j(\boldsymbol{r}) = 0$ by setting $r_j = j$.

Thirdly, we can exclude any price r_j with $(r_j - r_{j-1})^{\frac{1}{j-1}} > 1$. We show the irrelevance of these prices by $(r_j - r_{j-1})^{\frac{1}{j-1}} \geq 1 \geq \lambda \Rightarrow \max_{0 \leq k \leq j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} \lambda^i} \right\} \geq \frac{r_j - r_{j-1}}{\lambda^{j-1}} \geq 1 \geq \min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} \lambda^i}, 1 \right\} \Rightarrow p_j(\boldsymbol{r}) = \left(\min_{j+1 \leq k \leq c} \left\{ \frac{r_k - r_j}{\sum_{i=j}^{k-1} \lambda^i}, 1 \right\} - \max_{0 \leq k \leq j-1} \left\{ \frac{r_j - r_k}{\sum_{i=k}^{j-1} \lambda^i} \right\} \right)^+ = 0$. Thus, keeping r_j with $(r_j - r_{j-1})^{\frac{1}{j-1}} = 1$ in the action space is sufficient to enforce $p_j(\boldsymbol{r}) = 0$ if necessary.

Finally, we can exclude any price r_j with $(r_j - r_{j-1})^{\frac{1}{j-1}} > (r_{j+1} - r_j)^{\frac{1}{j}}$. The argumentation for this exclusion is covered by the proof of the following lemma:

Lemma 3 *Relevant prices r_j are given by* $\mathcal{R}_c = \left\{ \boldsymbol{r} \in \mathbb{R}^c : 0 \leq r_1 \leq \cdots \leq r_c \leq c,\ r_j \leq j\ \forall j, (r_j - r_{j-1})^{\frac{1}{j-1}} \leq 1\ for\ j \geq 2, and\ (r_j - r_{j-1})^{\frac{1}{j-1}} \leq (r_{j+1} - r_j)^{\frac{1}{j}}\ for\ 2 \leq j \leq c-1 \right\}$.

Proof: See Supplement S.3.

In Section 3.2, we have seen that $\underline{l}_k$ and $\bar{l}_k$ play an important role in calculating selling probabilities. With the action space reduction, we are now able to shed more light on the definition of these parameters.

Lemma 4 *It holds that* $\underline{l}_1(\boldsymbol{r}) = 0$, $\underline{l}_k(\boldsymbol{r}) = (r_k - r_{k-1})^{\frac{1}{k-1}} = \bar{l}_{k-1}(\boldsymbol{r})$, $2 \leq k \leq c$, $\bar{l}_c(\boldsymbol{r}) = 1$ *for all* $\boldsymbol{r} \in \mathcal{R}_c$.

Proof: See Supplement S.4.

With Lemma 4, we get closer to a closed-form expression of the probability function. However, we only have an implicit definition of $\underline{l}_{kj}(\boldsymbol{r})$ and $\bar{l}_{kj}(\boldsymbol{r})$ (cf. Section 3.2). We either define $\underline{l}_{kj}(\boldsymbol{r})$ and $\bar{l}_{kj}(\boldsymbol{r})$ such that $\int_{\underline{l}_{kj}(\boldsymbol{r})}^{\bar{l}_{kj}(\boldsymbol{r})} \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \, dl = 0$ or as intersection point between two curves.

The first case has no influence on the probability function and can be ignored. In the following, we will focus on the second case. By its definition as intersection point, it holds that for every (except one, cf. Remark 1) $\bar{l}_{kj}(\boldsymbol{r})$ there is one associated $\underline{l}_{ij}(\boldsymbol{r})$ such that $\bar{l}_{kj}(\boldsymbol{r}) = \underline{l}_{ij}(\boldsymbol{r})$ and $\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m} = \frac{r_i - r_j}{\sum_{m=j}^{i-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m}$. Moreover, it holds that small enough variations of the price vector at most change the place where both curves intersect. Particularly, small enough variations of r_m, $m \notin \{j, k, i\}$ do not change anything regarding the definition of $\bar{l}_{kj}(\boldsymbol{r})$ and corresponding $\underline{l}_{ij}(\boldsymbol{r})$.

Referring to the proof of Lemma 4, we can also add that $\underline{l}_j(\boldsymbol{r}) = \underline{l}_{j,j-1}(\boldsymbol{r})$ and $\underline{l}_{j+1}(\boldsymbol{r}) = \underline{l}_{j+1,j}(\boldsymbol{r})$. Again, small enough variations regarding the price vector at most change the place of these intersection points.

The following remark summarizes the observations above and will come in handy in the development of optimality conditions.

Remark 2 *It holds that:*

- *For every* $\bar{l}_{kj}(\boldsymbol{r}) \neq \bar{l}_j^{\text{end}}(\boldsymbol{r})$ *there is* $\underline{l}_{ij}(\boldsymbol{r})$ *such that* $\bar{l}_{kj}(\boldsymbol{r}) = \underline{l}_{ij}(\boldsymbol{r})$ *and* $\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m} = \frac{r_i - r_j}{\sum_{m=j}^{i-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m}$. *Moreover,* $\frac{d}{dr_m}\bar{l}_{kj}(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{ij}(\boldsymbol{r})$ for all m *and* $\frac{d}{dr_m}\bar{l}_{kj}(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{ij}(\boldsymbol{r}) = 0$ *for* $m \notin \{j, k, i\}$.
- $\underline{l}_{j+1}(\boldsymbol{r}) = \underline{l}_{j+1,j}(\boldsymbol{r})$ *with* $\frac{r_{j+1} - r_j}{\left(\underline{l}_{j+1}(\boldsymbol{r})\right)^j} = 1$. *Moreover,* $\frac{d}{dr_m}\underline{l}_{j+1}(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{j+1,j}(\boldsymbol{r})$ for all m *and* $\frac{d}{dr_m}\underline{l}_{j+1}(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{j+1,j}(\boldsymbol{r}) = 0$ *for* $m \notin \{j, j+1\}$.
- $\underline{l}_j(\boldsymbol{r}) = \underline{l}_{j,j-1}(\boldsymbol{r})$ *with* $\frac{r_j - r_{j-1}}{\left(\underline{l}_j(\boldsymbol{r})\right)^{j-1}} = 1$. *Moreover,* $\frac{d}{dr_m}\underline{l}_j(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{j,j-1}(\boldsymbol{r})$ for all m *and* $\frac{d}{dr_m}\underline{l}_j(\boldsymbol{r}) = \frac{d}{dr_m}\underline{l}_{j,j-1}(\boldsymbol{r}) = 0$ *for* $m \notin \{j-1, j\}$.

4.2 Optimality conditions

With the previous section, we gathered enough information regarding the probability function to advance to our main goal: optimizing value function (11).

Before we engage the partial differentiation of the value function, we first want to elaborate more on the partial differentiation of probability function (5). The calculation of $\frac{d}{d\,r_i} p_j(\boldsymbol{r})$ varies a little depending on the following three cases: $i > j$, $i < j$, and $i = j$.

Lemma 5 *It holds:*

1. *For* $i > j$, $\frac{d}{d\,r_i} p_j(\boldsymbol{r}) = \int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})} \frac{1}{\sum_{m=j}^{i-1} l^m}\, dl$
2. *For* $i < j$, $\frac{d}{d\,r_i} p_j(\boldsymbol{r}) = \int_{\underline{l}_{ji}(\boldsymbol{r})}^{\bar{l}_{ji}(\boldsymbol{r})} \frac{1}{\sum_{m=i}^{j-1} l^m}\, dl$
3. *For* $i = j$, $\frac{d}{d\,r_i} p_j(\boldsymbol{r}) = -\sum_{k=i+1}^{c} \int_{\underline{l}_{ki}(\boldsymbol{r})}^{\bar{l}_{ki}(\boldsymbol{r})} \frac{1}{\sum_{m=i}^{k-1} l^m}\, dl - \sum_{k=0}^{i-1} \int_{\underline{l}_{ik}(\boldsymbol{r})}^{\bar{l}_{ik}(\boldsymbol{r})} \frac{1}{\sum_{m=k}^{i-1} l^m}\, dl$

Proof: See Supplement S.5.

With the additional knowledge about the probability function, we now can turn our focus on the first-order condition. Therefore, we calculate the partial differentiation of value function (11):

$$\frac{d}{d\,r_i}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right) = p_i(\boldsymbol{r}) + \sum_{j=1}^{c}\left(\frac{d}{d\,r_i} p_j(\boldsymbol{r})\right) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)$$

$$= p_i(\boldsymbol{r}) + \sum_{j=1}^{i-1} \int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})} \frac{1}{\sum_{m=j}^{i-1} l^m}\, dl \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)$$

$$+ \left(-\sum_{k=i+1}^{c} \int_{\underline{l}_{ki}(\boldsymbol{r})}^{\bar{l}_{ki}(\boldsymbol{r})} \frac{1}{\sum_{m=i}^{k-1} l^m}\, dl - \sum_{k=0}^{i-1} \int_{\underline{l}_{ik}(\boldsymbol{r})}^{\bar{l}_{ik}(\boldsymbol{r})} \frac{1}{\sum_{m=k}^{i-1} l^m}\, dl\right) \cdot \left(r_i - \Delta_i V_{t-1}(c)\right)$$

$$+ \sum_{j=i+1}^{c} \int_{\underline{l}_{ji}(\boldsymbol{r})}^{\bar{l}_{ji}(\boldsymbol{r})} \frac{1}{\sum_{m=i}^{j-1} l^m}\, dl \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)$$

$$= p_i(\boldsymbol{r}) + p_i(\boldsymbol{r}) - \int_{\underline{l}_i(\boldsymbol{r})}^{\bar{l}_i(\boldsymbol{r})} 1\, dl - \sum_{k=i+1}^{c} \int_{\underline{l}_{ki}(\boldsymbol{r})}^{\bar{l}_{ki}(\boldsymbol{r})} \frac{\Delta_k V_{t-1}(c) - \Delta_i V_{t-1}(c)}{\sum_{m=i}^{k-1} l^m}\, dl$$

$$+ \sum_{k=0}^{i-1} \int_{\underline{l}_{ik}(\boldsymbol{r})}^{\bar{l}_{ik}(\boldsymbol{r})} \frac{\Delta_i V_{t-1}(c) - \Delta_k V_{t-1}(c)}{\sum_{m=k}^{i-1} l^m}\, dl$$

An optimal solution for (11) is either on the boundary of $\mathcal{R}_c$ or it fulfills $\frac{d}{d\,r_i}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right) = 0$ for every i.

While the optimality condition is clear for an interior point, we want to briefly investigate optimality on the boundary: Every solution on the boundary of $\mathcal{R}_c$ is characterized by having at least one batch size j

with $p_j(\boldsymbol{r}) = 0$. For the following argumentation, we assume that there is only one such j. However, we could repeatedly apply the coming arguments for cases where demand for several batch sizes is eliminated.

By simply removing the option to purchase j units beforehand and repeating the whole optimization, we technically get the same optimality condition as above. The only difference is that the sum skips index j. However, we could add this index back to the sum as, by definition, $\underline{l}_{ji}(\boldsymbol{r}) = \bar{l}_{ji}(\boldsymbol{r})$ ($\underline{l}_{ij}(\boldsymbol{r}) = \bar{l}_{ij}(\boldsymbol{r})$) for $j > i$ ($j < i$), and thus, $\int_{\underline{l}_{ji}(\boldsymbol{r})}^{\bar{l}_{ji}(\boldsymbol{r})} \frac{\Delta_j V_{t-1}(c) - \Delta_i V_{t-1}(c)}{\sum_{m=i}^{j-1} l^m} \, dl = 0$ ($\int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})} \frac{\Delta_i V_{t-1}(c) - \Delta_j V_{t-1}(c)}{\sum_{m=j}^{i-1} l^m} \, dl = 0$). Again, based on the definition of $\underline{l}_j(\boldsymbol{r})$, $\bar{l}_j(\boldsymbol{r})$, $\underline{l}_{kj}(\boldsymbol{r})$, $\bar{l}_{kj}(\boldsymbol{r})$, $\underline{l}_{jk}(\boldsymbol{r})$, and $\bar{l}_{jk}(\boldsymbol{r})$, it holds that $\int_{\underline{l}_j(\boldsymbol{r})}^{\bar{l}_j(\boldsymbol{r})} 1 \, dl + \sum_{k=j+1}^{c} \int_{\underline{l}_{kj}(\boldsymbol{r})}^{\bar{l}_{kj}(\boldsymbol{r})} \frac{\Delta_k V_{t-1}(c) - \Delta_j V_{t-1}(c)}{\sum_{m=j}^{k-1} l^m} \, dl - \sum_{k=0}^{j-1} \int_{\underline{l}_{jk}(\boldsymbol{r})}^{\bar{l}_{jk}(\boldsymbol{r})} \frac{\Delta_j V_{t-1}(c) - \Delta_k V_{t-1}(c)}{\sum_{m=k}^{i-1} l^m} \, dl = 0$ if $p_j(\boldsymbol{r}) = 0$. All in all, we can conclude that the optimal solution has to fulfill $\frac{d}{d\, r_i}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right) = 0$ regardless of being on the boundary or an interior point.

Proposition 1 *The optimal solution of (11) meets for every batch size* i *the following condition:*

$$p_i(\boldsymbol{r}) = \frac{1}{2}\left(\underline{l}_{i+1}(\boldsymbol{r}) - \underline{l}_i(\boldsymbol{r}) + \sum_{k=i+1}^{c} \int_{\underline{l}_{ki}(\boldsymbol{r})}^{\bar{l}_{ki}(\boldsymbol{r})} \frac{\Delta_k V_{t-1}(c) - \Delta_i V_{t-1}(c)}{\sum_{m=i}^{k-1} l^m} \, dl - \sum_{k=0}^{i-1} \int_{\underline{l}_{ik}(\boldsymbol{r})}^{\bar{l}_{ik}(\boldsymbol{r})} \frac{\Delta_i V_{t-1}(c) - \Delta_k V_{t-1}(c)}{\sum_{m=k}^{i-1} l^m} \, dl\right)$$

Proof: The proof can be found above the proposition.

With this optimality condition, we can calculate the optimal probability to sell at least one unit. Thereby, we can make the following observation.

Remark 3 *For the optimal solution* $\boldsymbol{r}^*$ *of (11), it holds that* $\sum_{i=1}^{c} p_i(\boldsymbol{r}^*) = \frac{1}{2} - \sum_{k=1}^{c} \int_{\underline{l}_{k0}(\boldsymbol{r}^*)}^{\bar{l}_{k0}(\boldsymbol{r}^*)} \frac{\Delta_k V_{t-1}(c)}{\sum_{m=0}^{k-1} l^m} \, dl$. *Thus, the overall selling probability is less than or equal to* 0.5 *and decreasing in opportunity costs.*

In states with $t = 1$ there are no opportunity costs. There, the optimality condition simplifies to $p_i(\boldsymbol{r}) = \frac{1}{2}\left(\underline{l}_{i+1}(\boldsymbol{r}) - \underline{l}_i(\boldsymbol{r})\right)$. Moreover, the optimal overall selling probability is 0.5.

Proposition 2 *If* $\Delta_k V_{t-1}(c) = 0$ *for every* k, *the optimal solution for (11) is an interior point of* $\mathcal{R}_c$ *and fulfills* $p_i(\boldsymbol{r}) = \frac{1}{2}\left(\underline{l}_{i+1}(\boldsymbol{r}) - \underline{l}_i(\boldsymbol{r})\right)$.

Proof: We will proof $p_k(\boldsymbol{r}) \neq 0$ for every k by contradiction. Therefore, we make use of the observation that $p_k(\boldsymbol{r}) \neq 0$ is equivalent to $\underline{l}_k(\boldsymbol{r}) < \underline{l}_{k+1}(\boldsymbol{r})$.

We assume that for the optimal solution $\boldsymbol{r}^* \in \mathcal{R}_c$ there is one batch size k with $p_k(\boldsymbol{r}^*) = 0$. In the following, we will focus on a proof with only one such k. However, the technique we use in this proof could repeatedly applied (with small adjustments) to contradict optimal solutions with several batch sizes k such that $p_k(\boldsymbol{r}^*) = 0$.

In a first step, we modify the optimal solution by slightly decreasing r_k^* to $r_k^* - \epsilon$ with $\epsilon > 0$ small enough. We denote this modified version by $\boldsymbol{r}^{*,\epsilon}$. With $\boldsymbol{r}^{*,\epsilon}$, the probability to sell a batch of size k is positive as $\underline{l}_k(\boldsymbol{r}^{*,\epsilon}) < \underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})$.

<u>$k > 1$</u>:

We recall the argumentation used in Lemma 4 to conclude that $p_k(\boldsymbol{r}^{*,\epsilon}) = \left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right) + \int_{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{l^k}\, dl - \int_{\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}}{l^{k-1}}\, dl$ with $\bar{l}(\boldsymbol{r}^{*,\epsilon}) = \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}} = \frac{r_{k+1}^* - r_k^* + \epsilon}{r_k^* - \epsilon - r_{k-1}^*}$. This $\bar{l}(\boldsymbol{r}^{*,\epsilon})$ exists for every ϵ that is small enough.

Plucking $\boldsymbol{r}^{*,\epsilon}$ into the first partial deviation leads to:

$$\frac{d}{dr_k}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right)\Bigg|_{\boldsymbol{r}=\boldsymbol{r}^{*,\epsilon}} = 2 \cdot p_k(\boldsymbol{r}^{*,\epsilon}) - \left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right)$$

$$= 2 \cdot \left(\left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right) + \int\limits_{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{l^k}\, dl - \int\limits_{\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}}{l^{k-1}}\, dl \right)$$

$$- \left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right)$$

As $1 \geq \frac{\left(r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}\right)^k}{\left(r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}\right)^{k-1}} \geq \frac{1}{2}$ for ϵ small enough, it holds that $\left(\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})\right) + \int_{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_{k+1}^{*,\epsilon} - r_k^{*,\epsilon}}{l^k}\, dl - \int_{\underline{l}_k(\boldsymbol{r}^{*,\epsilon})}^{\bar{l}(\boldsymbol{r}^{*,\epsilon})} \frac{r_k^{*,\epsilon} - r_{k-1}^{*,\epsilon}}{l^{k-1}}\, dl < \frac{\underline{l}_{k+1}(\boldsymbol{r}^{*,\epsilon}) - \underline{l}_k(\boldsymbol{r}^{*,\epsilon})}{2}$. Together, it holds that $\frac{d}{dr_k}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right)\Big|_{\boldsymbol{r}=\boldsymbol{r}^{*,\epsilon}} < 0$.

Based on the negativity of this partial deviation, the value function is strictly decreasing for every $\boldsymbol{r}^{*,\epsilon}$ (with $\epsilon > 0$ small enough). Thus, in contradiction to the optimality of $\boldsymbol{r}^*$, any $\boldsymbol{r}^{*,\epsilon}$ yields higher expected revenues than $\boldsymbol{r}^*$. A similar argumentation works for r_c.

<u>$k = 1$:</u>

If $p_1(\boldsymbol{r}^*) = 0$, it follows that $r_1^* = r_2^*$. To show the suboptimality of this condition, we need to examine two cases: $r_2^* > \exp(-0.5)$ and $r_2^* \leq \exp(-0.5)$. First, we assume that $r_2^* > \exp(-0.5)$.

By definition of $p_1(\boldsymbol{r}^{*,\epsilon})$ (cf. (5)), $p_1(\boldsymbol{r}^{*,\epsilon}) = (r_2^{*,\epsilon} - r_1^{*,\epsilon}) + \int_{r_2^{*,\epsilon} - r_1^{*,\epsilon}}^{\frac{r_2^{*,\epsilon} - r_1^{*,\epsilon}}{r_1^{*,\epsilon}}} \frac{r_2^{*,\epsilon} - r_1^{*,\epsilon}}{l}\, dl - \int_0^{\frac{r_2^{*,\epsilon} - r_1^{*,\epsilon}}{r_1^{*,\epsilon}}} r_1^{*,\epsilon}\, dl$. Solving the integrals leads to $p_1(\boldsymbol{r}^{*,\epsilon}) = -\log(r_1^{*,\epsilon}) \cdot (r_2^{*,\epsilon} - r_1^{*,\epsilon})$. With this inequality, it holds that $\frac{d}{dr_k}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right)\Big|_{\boldsymbol{r}=\boldsymbol{r}^{*,\epsilon}} < 0$ for $r_1^{*,\epsilon} \in (\exp(-0.5), r_2^{*,\epsilon})$. From here on, the same argumentation as above applies.

Next, we assume that $r_2^{*,\epsilon} \leq \exp(-0.5)$. As every customer with $w \cdot (1 + l) \geq r_2^{*,\epsilon}$ is purchasing at least 2 units, we can conclude that $\sum_{j=1}^{c} p_j(\boldsymbol{r}^{*,\epsilon}) \geq 1 - \int_0^1 \frac{r_2^{*,\epsilon}}{1+l} dl = 1 - r_2^{*,\epsilon} \cdot (\log(2) - \log(1)) \geq 1 - \exp(-0.5) \cdot \log(2) > 0.5$. Since the optimal solution must fulfill the condition $\sum_{j=1}^{c} p_j(\boldsymbol{r}) = 0.5$ (cf. Remark 3), $r_2^{*,\epsilon} \leq \exp(-0.5)$ cannot occur and has no relevance for our proof.

All in all, we have shown that $\boldsymbol{r}$ with r_k such that $p_k(\boldsymbol{r}) = 0$ cannot be an optimal solution. This ultimately leads to the statement that the optimal solution is an interior point of $\mathcal{R}_c$. □

Even though we developed the optimality conditions, finding all solutions in every state that fulfill these equations is a difficult task. We are facing a system with c nonlinear equations that are additionally plagued by integrals and implicitly defined variables. Most of these difficulties arise from the analytical intractability of the choice model. Thus, online solving (11) to optimality in every state is out of scope.

4.3 Fluid approximation and approximate optimality

Approximate optimality is a desirable property to ensure theoretical performance guarantees. In literature (see, e.g., Maglaras and Meissner 2006), this is often done based on the solution of a deterministic (fluid) model.

This section is built as follows: We start with formulating the deterministic (fluid) model and discussing approximate optimality of its solution. Because of the analytical intractability induced by our choice model, we introduce an altered model that constitutes an upper bound of our fluid model and is easier to solve. At the end of this section, we show that the optimal solution of the altered model is also an optimal solution of our fluid model with the same objective value.

4.3.1. Problem formulation

In the fluid model, we assume that capacity is continuously splitable, and demand is no longer given by stochastic progress. It is rather deterministically determined by its rate and, thus, we assume that customers purchase continuous fractions of batches.

We denote deterministic continuous demand for batch j at time t by $p_j^d(\boldsymbol{r}_t)$ with $p_j^d(\boldsymbol{r}_t) = p_k(\boldsymbol{r}_t)$ and capacity is depleted by $\sum_{j=1}^{C} j \cdot p_j^d(\boldsymbol{r}_t)$ at time t. The fluid model is given by:

$$\max_{\boldsymbol{r}_t \in \mathcal{R}_c \, \forall t} \left\{ \sum_{t=1}^{T} \sum_{j=1}^{C} r_{jt} \cdot p_j^d(\boldsymbol{r}_t) \, : \, \sum_{t=1}^{T} \sum_{j=1}^{C} j \cdot p_j^d(\boldsymbol{r}_t) \leq C \; and \; \boldsymbol{r}_t \geq 0 \right\}$$

With a time-homogeneous demand function, one can easily verify that the optimization problem above is equal to:

$$\max_{\boldsymbol{r} \in \mathcal{R}_c} \left\{ T \cdot \sum_{j=1}^{C} r_j \cdot p_j^d(\boldsymbol{r}) \, : \, \sum_{j=1}^{C} j \cdot p_j^d(\boldsymbol{r}) \leq \frac{C}{T} \right\} \tag{12}$$

By equality of both problem formulations, a static pricing policy is optimal for the fluid model. Based on a similar structure, our fluid model (12) can be brought into the form of the multiproduct fluid model examined by Maglaras and Meissner (2006). Similarly, their proof regarding approximate optimality also holds for the static pricing policy received by (12). Applying this static policy in our dynamic setting is therefore approximately optimal in a regime where demand and capacity grow proportionally large. This definition is called the first-order asymptotic optimality criterion by Gallego and van Ryzin (1997) and Cooper (2002). Moreover, Maglaras and Meissner (2006) show that resolving the fluid policy throughout the selling horizon is again approximately optimal.

Remark 4 *The optimal solution of the unrestricted version of (12) is equivalent to the optimal solution of the dynamic model (11) in states* $t = 1$. *Thereby, Proposition 2 also defines the optimal solution for this case. If this solution does not fulfill* $\sum_{j=1}^{C} j \cdot p_j^d(\boldsymbol{r}) \leq \frac{C}{T}$, *we still can use* $\frac{d}{d\, r_i}\left(\sum_{j=1}^{c} p_j(\boldsymbol{r}) \cdot \left(r_j - \Delta_j V_{t-1}(c)\right)\right) = p_i(\boldsymbol{r}) - \frac{1}{2}\left(\underline{l}_{i+1}(\boldsymbol{r}) - \underline{l}_i(\boldsymbol{r})\right)$ *to find the optimal solution.*

4.3.2. Altered fluid model

The fluid model above still suffers from the analytically intractability induced by our choice model. To overcome this difficulty, we introduce an altered version of the fluid model. The difference between the original fluid model and the altered version results from a simplification of the choice model.

Instead of calculating the probability that $u_k(\boldsymbol{r}) = \max_{j=0,\dots,C}\{u_j(\boldsymbol{r})\}$, we now calculate the probability that $u_k(\boldsymbol{r}) = \max_{j=0,k-1,k,k+1}\{u_j(\boldsymbol{r})\}$. Thereby, we reduce competition between available options and, thus, only compare three instead of $c - 1$ options. We denote demand by this altered choice model as $p_j^{d,a}(\boldsymbol{r})$ and write $p_j^{d,a}(\boldsymbol{r}) = \mathbb{P}\left(u_j(\boldsymbol{r}) = \max_{k=0,j-1,j,j+1}\{u_k(\boldsymbol{r})\}\right)$. Technically, through the alternation, the choice model is no longer a choice model. By summing up all demand rates, we get a value greater than or equal to 1, i.e. $\sum_{j=0}^{C} p_j^{d,a}(\boldsymbol{r}) \geq 1$.

The optimal solution of this altered model is an upper bound to (12). This can be verified as follows: The optimal solution $\boldsymbol{r}^*$ of (12) is either feasible for the altered model or $\sum_{j=1}^{C} j \cdot p_j^{d,a}(\boldsymbol{r}) > \frac{C}{T}$. If it is feasible, the statement is obviously true considering $p_j^{d,a}(\boldsymbol{r}) \geq p_j^d(\boldsymbol{r}^*)$. In the letter case, we can increase $\boldsymbol{r}^*$ in such a way to $\boldsymbol{r}^{**}$ that $p_j^{d,a}(\boldsymbol{r}^{**}) = p_j^d(\boldsymbol{r}^*)$. Thereby, we impose the same demands with higher prices.

Similar considerations as in Section 4.2 lead to the following remark.

Remark 5 *With* $p_j^{d,a}(\boldsymbol{r})$, *the statement of Remark 4 holds analogously.*

We introduced this modification to reduce complexity and to derive a choice model we can better handle (see Supplement S.7 for a calculation of $p_j^{d,a}(\boldsymbol{r})$). Indeed, we are now able to (numerically) calculate the optimal solution $\boldsymbol{r}^*$ of the altered fluid model.

However, calculating $\boldsymbol{r}^*$ is only the first step. The second step is verifying that this solution is also optimal for the fluid model. We have already discussed that the altered fluid model is an upper bound.

If we can verify that $\boldsymbol{r}^*$ is feasible for the fluid model and results in the same revenues, we showed optimality of $\boldsymbol{r}^*$ for the fluid model. Thereby, it suffices to check if $p_j^{d,a}(\boldsymbol{r}^*) = p_j^d(\boldsymbol{r}^*)$. Instead of calculating $p_j^d(\boldsymbol{r}^*)$ for every j, we developed the following, easy verifiable conditions.

Lemma 6 *If* $\bar{l}_j^{end}(\boldsymbol{r}) < \frac{r_{k+1}-r_k}{r_k-r_{k-1}}$ *for every* j, k *with* $2 \leq k \leq j$, $p_j^{d,a}(\boldsymbol{r}) = p_j^d(\boldsymbol{r})$ *for every* j.

Proof: See Supplement S.6.

In our numerical study, we calculated the optimal solution for the altered fluid model for every combination of $T \leq 10$ and $C \leq 20$. The condition of Lemma 6 was always fulfilled. Therefore, in these cases, this solution could be used as a static pricing policy that is approximately optimal in our dynamic setting. The same holds for a policy that periodically solves the altered fluid model with actualized capacity and time-to-go. Moreover, every policy that leads to higher expected revenues at every stage of the optimization problem is again approximately optimal.

5 Solution methods for the general setting

In this section, we construct three heuristics for the general setting to solve the problem regardless of its analytical intractability. Two of these approaches are based on the results of Schur (2023) and use the optimal solution in a setting where the firm has access to customers' private information, i.e., their base willingness-to-pay and their consumption indicator, respectively. The third approach can be described as a decomposition in units. Thereby, we allow customers to buy the jth unit of the product without buying the units $1, 2, \ldots, j-1$. Even though this makes no sense in terms of practical applicability, it makes the optimization problem much easier and allows us to solve this new problem. Based on this solution, we can then develop batch prices for our original problem.

In the previous section, we have seen that the solution of a fluid approximation is approximately optimal. As we want to ensure this property in the following heuristics, we check in every state which solution performs best: the one derived by our mechanism or the solution of the fluid approximation. The better-performing one is finally adopted by the heuristic.

5.1 Approaches 1 and 2: Expected optimal batch prices

With the results of Schur (2023), we can build two of our heuristics. The main idea is that optimal batch prices $r_{jt}(c|w)$ and $r_{jt}(c|l)$ are known if the firm can observe the realization of random base willingness-to-pay and consumption indicator, respectively. In these cases, the optimal batch prices depend on realizations of random variables, and thus, are themself random variables. Thus, we can calculate expected optimal batch prices and use them as a proxy for the optimal solution of optimization problem (9).

Both approaches consist of the same steps with the only difference being underlying (realization-dependent) optimal batch prices: $r_{jt}(c|w)$ (Approach 1) and $r_{jt}(c|l)$ (Approach 2). After showing the definition of these batch prices, we will explain the remaining steps without distinguishing between Approaches 1 and 2.

While $r_{jt}(c|l)=\frac{1}{2}\left(\sum_{k=0}^{j-1} l^k+\Delta_j V_{t-1}^E(c)\right)$, with $j\leq N_t(c|l)=\max_{j=1,\ldots,c}\{j\colon \Delta_1 V_{t-1}^E(c-j+1)<l^{j-1}\}$, is given in closed form, $r_{jt}(c|w)$ is implicitly defined by $w\cdot\left(\frac{r_{jt}(c|w)}{w}\right)^{\frac{j-2}{j-1}}\cdot\left((j-1)-j\cdot\left(\frac{r_{jt}(c|w)}{w}\right)^{\frac{1}{j-1}}\right)+\Delta_1 V_{t-1}^E(c+1-j)=0$, with $j\leq N_t(c|w)=\max_{j=1,\ldots,c}\{j\colon \Delta_1 V_{t-1}^E(c-j+1)<w\}$.

With these realizations of optimal batch prices, we start our heuristics by building expected optimal batch prices:

$$r_{jt}^E(c)=\frac{\int_0^1 r_{jt}(c|x)\cdot 1_{\{j<N_t(c|x)\}}\,dx}{\int_0^1 1_{\{j<N_t(c|x)\}}\,dx}\qquad \text{for } j=1,\ldots,c, \tag{13}$$

Precisely, we are calculating conditional expected optimal batch prices where we only take realizations of λ and ω, respectively, into account that can lead to possible economic sales, i.e. $r_{j+1,t}(c|\cdot)-r_{jt}(c|\cdot)\geq\Delta_1 V_{t-1}^E(c+1-j)$. Other realizations are of no interest as there is no clear pricing strategy except pricing the batch out which can be achieved by arbitrary batch prices that are high enough, i.e. $r_{jt}(c|l)\geq\sum_{k=0}^{j-1} l^k$ and $r_{jt}(c|w)\geq w$. Thus, these events where we did not plan to sell anything should not be used to find overall good batch prices.

As we have a closed-form expression of $r_{jt}(c|l)$, we can rewrite (13) to $r_{jt}^{E(\lambda)}(c)=0.5\cdot\left(\Delta_j V_{t-1}^{E(\lambda)}(c)+\frac{1}{1-\left(\Delta_1 V_{t-1}^{E(\lambda)}(c+1-j)\right)^{\frac{1}{j-1}}}\cdot\sum_{k=0}^{j-1}\frac{1-\left(\Delta_1 V_{t-1}^{E(\lambda)}(c+1-j)\right)^{\frac{k+1}{j-1}}}{k+1}\right)$ for Approach 2. The same is not possible for Approach 1, where we resort to numerically calculating $r_{jt}(c|w)$ for as many realizations w as possible to accurately derive $r_{jt}^{E(\omega)}(c)$.

We compare expected revenue-to-go derived by expected optimal batch prices $r_{jt}^E(c)$ and the solution of the fluid approximation $r_{jt}^{FA}(c)$:

$$V_t^E(c)=\max_{r\in\{r_t^E(c),r_t^{FA}(c)\}}\left\{\sum_{j=1}^c p_j(\boldsymbol{r})\cdot\left(r_j+V_{t-1}^E(c-j)\right)+\left(1-\sum_{j=1}^c p_{jt}(\boldsymbol{r})\right)\cdot V_{t-1}^E(c)\right\}, \tag{14}$$

with the same boundary conditions as the original problem (9). The better-performing batch prices are then adopted by our heuristics.

Remark 6 *By using suboptimal batch prices, we get a lower bound to optimization problem (9). Thus, it holds that $V_t(c)\geq V_t^E(c)$ for every (t,c).*

We sum up the first two heuristics that are based upon the idea of expected optimal batch prices by the following pseudo code:

Pseudo code Approaches 1 and 2: Expected optimal batch prices

Input: time horizon T, starting stock C

Output: batch prices for every batch size j and state (t, c): $r_{jt}^{E}(c)$

1. Initialize value functions $V_0^E(c) = 0$ for every $c = 0, 1, \dots, C$ and $V_t^E(0) = 0$ for every $t = 0, 1, \dots, T$ ▷ initialize boundary conditions

2. For $t = 1, 2, \dots, T$ **do** ▷ loop over time horizon

2.1. For $c = 1, 2, \dots, C$ **do** ▷ loop over capacity

2.1.1. Calculate $r_{jt}^{E}(c)$ for every $j = 1, 2, \dots, c$ ▷ calculate expected optimal batch prices

2.1.2. Calculate $V_t^E(c)$ with (14) ▷ calculate expected revenue-to-go

2.1.3. Save maximizing price vector of (14) as approximately optimal solution

5.2 Approach 3: Decomposition in units

Our next algorithm is based on a decomposition approach. The basic idea is that we allow customers to buy the jth unit of the product even though they might not buy units 1 to $j - 1$. As every unit of the product is the same, there is no distinction between the $1st$, 2nd or jth unit other than the number a customer already has in her or his basket. Thus, this decomposition is merely theoretical without immediate practical applicability. However, it greatly simplifies the optimization problem. A hypothetical customer now faces c distinct binary decisions instead of one decision with $c + 1$ options. This, in turn, enables the firm to solve c distinct and rather simple optimization problems instead of one complex problem. With this simplification, we can derive batch prices that can be used as proxy for the optimal batch prices to optimization problem (9).

To consider this decomposition, we must change the customer choice model. Customers still strive to maximize their utility. But, instead of purchasing j units if and only if $u_j(\boldsymbol{r}) = \max_{j=0,\dots,c}\{u_j(\boldsymbol{r})\}$ with $u_0(\boldsymbol{r}) = 0$ denoting the no-purchase option, they decide for every single unit whether they want to purchase it or not. This decision is based upon whether the additional willingness-to-pay for the jth unit is at least as high as the additional price the customer has to pay, i.e. $X_j - X_{j-1} = \omega \cdot (\lambda)^{j-1} \geq r_j - r_{j-1}$. If the customer decides to purchase the jth unit, she or he must pay $r_j - r_{j-1}$. For example, for given batch prices, a customer might only be willing to purchase the second and fourth unit due to the willingness-to-pay curve. In this case, the customer pays $(r_2 - r_1) + (r_4 - r_3)$ to get 2 units of the product in total.

By allowing this decomposition, customers' decision problem changes. The model itself becomes easier as the decision between several options is split in several simple decisions. This makes anticipating customers' behavior easier. The firm no longer must decide what price vector containing all batch prices it should quote and how the customer reacts to this vector. Now, the firm can split its one decision in c decisions where it each has only to decide about the price of the jth unit while anticipating how likely the customer would buy the jth unit (which happens solely based on the price of the jth unit). Therefore, the decision variable becomes the price of the jth unit, $r^{D,j}$. As $r^{D,j} = r_j - r_{j-1}$, introducing $r^{D,j}$ is merely a change in notation. The optimization problem we are now focusing on is given by:

$$\sum_{j=1}^{c} \left(\max_{r^{D,j} \geq 0} \left\{ p_t^{D,j}\left(r^{D,j}\right) \cdot \left(r^{D,j} - \Delta_1 V_{t-1}^{D}\left(c - (j-1)\right)\right) \right\} \right) + V_{t-1}^{D}(c), \tag{15}$$

with $p_t^{D,j}\left(r^{D,j}\right)$ denoting the probability that a customer is purchasing the jth unit of the product and $V_{t-1}^{D}(c)$ denoting the expected revenue-to-go.

Probability $p_t^{D,j}\left(r^{D,j}\right)$ only depends on $r^{D,j}$ and can be calculated by $p_t^{D,j}\left(r^{D,j}\right) = \int_0^1 \int_0^1 1_{\{w \cdot (l)^{j-1} \geq r^{D,j}\}}(w,l)\, dw\, dl = 1 - \frac{j-1}{j-2}\left(r^{D,j}\right)^{\frac{1}{j-1}} + \frac{r^{D,j}}{j-2}$. As the new optimization problem (15) is a sum over c singleunit dynamic pricing problems, it becomes an easy to solve optimization problem. With $r_t^{D,j}(c)$ denoting the optimal solution to (15), we can use $r_{jt}^{D}(c) = \sum_{i=1}^{j} r_t^{D,i}(c)$ as a proxy of the optimal batch prices for the original optimization problem (9) and can evaluate expected revenue-to-go derived by $r_{jt}^{D}(c)$ and by $r_{jt}^{FA}(c)$:

$$V_t^D(c) = \max_{\boldsymbol{r} \in \{\boldsymbol{r}_t^D(c), \boldsymbol{r}_t^{FA}(c)\}} \left\{ \sum_{j=1}^{c} p_{jt}(\boldsymbol{r}) \cdot \left(r_j + V_{t-1}^{D}(c-j)\right) + \left(1 - \sum_{j=1}^{c} p_{jt}(\boldsymbol{r})\right) \cdot V_{t-1}^{D}(c) \right\}. \tag{16}$$

The boundary conditions are again given by $V_0^D(c) = 0$ for $c \geq 0$ and $V_t^D(0) = 0$ for $t \geq 0$.

Remark 7 *By using suboptimal batch prices, we get a lower bound to optimization problem (9). Thus, it holds that* $V_t(c) \geq V_t^D(c)$ *for every* (t,c).

The heuristic can be summed up by the following pseudo code:

Pseudo code Approach 2: Decomposition of units

Input: time horizon T, starting stock C

Output: batch prices for every batch size j and state (t,c): $r_{jt}^{D}(c)$

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1. Initialize value functions $V_0^D(c) = 0$ for every $c = 0, 1, \dots, C$ and $V_t^D(0) = 0$ for every $t = 0, 1, \dots, T$ ▷ initialize boundary conditions

2. **For** $t = 1, 2, \dots, T$ **do** ▷ loop over time horizon

2.1. **For** $c = 1, 2, \dots, C$ **do** ▷ loop over capacity

2.1.1. Solve optimization problem (15) to get $r_t^{D,j}(c)$ for every $j = 1, 2, \dots, c$ ▷ calculate optimal price for the jth unit for every j

2.1.2. Calculate $r_{jt}^{D}(c) = \sum_{i=1}^{j} r_t^{D,i}(c)$ for every $j = 1, 2, \dots, c$ ▷ calculate proxy for optimal batch prices

2.1.3. Calculate $V_{t-1}^{D}(c)$ with (16) ▷ calculate expected revenue-to-go

2.1.4. Save maximizing price vector of (16) as approximately optimal solution

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6 Numerical studies

In this section, we focus on the general setting and examine the performance of all heuristics mentioned in the previous section. Nevertheless, we also implemented and simulated the special case *PI* where the firm can observe next customers' consumption indicator. Thereby, we get an upper bound for our unknown optimal solution and can compare our heuristics against it.

To evaluate our heuristics, we implemented the following mechanisms:

- $E(\lambda)$ and $E(\omega)$ are mechanisms to approximately solve (9) based on the idea of expected optimal batch prices (Section 5.1).
- D is a mechanism to approximately solve (9) based on decomposition in units (Section 5.2).
- S is a mechanism that solves a standard singleunit dynamic pricing problem and, thereby, ignores the fact that customers might be willing to purchase more than just one unit. As a singleunit dynamic pricing procedure results in a price for only one unit of the product, we then extend it in a linear matter to get batch prices.
- L is a mechanism that solves (9) numerically but with an additional constraint that restricts the batch prices to a linear pricing scheme, i.e. $r_j = j \cdot r_1$ for every j. Thus, the optimization problem simplifies as we only have one decision variable.
- *PI* is a mechanism where we observe customers' consumption indicator before quoting batch prices; the corresponding optimization model is given in Schur (2023). This mechanism sets optimal batch prices in a setting where the firm has additional information, and thus, yields an upper bound to our setting.
- PI^{lin} is a mechanism where we solve optimization problem *PI* numerically while restricting the action space to linear batch prices, only. This mechanism, in combination with *PI* and *L*, is used to find an approximation to our original problem.

Besides our heuristics, we chose *S* and *L* for our numerical studies, as it seems natural that these are the mechanisms that might be applied the most in practice. Mechanism *S* is conducting standard dynamic pricing and, thus, ignoring multiunit demand. Firms that are aware of customers purchasing more than just one unit might apply *L* as this is the obvious choice without specialized optimization problems that merely exist in literature. The last two mechanisms, *PI* and PI^{lin}, are mainly there to test heuristics later against an upper bound and an approximation of the (unknown) value of the objective function (9). However, we should keep in mind that *PI* determines the revenue that could be earned if the firm has additional information about customers' preferences, and thus, shows the inherent advantage of this special case in comparison to our general case.

Every mechanism provides a policy that contains batch prices. To evaluate the mechanisms, we perform simulation studies. Therefore, we generated 10,000 customer streams in advance and applied every policy we derived from the mechanisms to these streams separately. One simulation run consists of a complete sales process containing T specific customers and a batch price quoted in every period (depending on the capacity left) according to the mechanism investigated. After observing the decision of the current customer, a new batch price is set for the next customer. Repeating this procedure until end of selling horizon leads to a total revenue for this simulation run. 10,000 simulation runs are leading to 10,000 total revenues we are taking the mean of. As this is done for every mechanism on the same 10,000 customer streams, the simulated revenues can be immediately compared to each other. We did these simulation runs for 20 different settings with $T = 10$ and $C = 1, \dots, 20$.

In the following, we show simulated revenues of all mechanism and compare them to the upper bound received from special case PI and an approximation of the (unknown) value function (9). Thereby, we can observe that heuristics D and $E(\omega)$ are resulting in the highest revenues, while $E(\lambda)$ is slightly behind. We then examine the selling strategies of all heuristics by analyzing simulation results. Moreover, we discuss several evolutions of batch prices of both best performing heuristics, D and $E(\omega)$. As we thereby observed a piecewise-linear pattern in batch prices, we finally conducted another numerical study to evaluate a piecewise-linear pricing scheme. This study shows that such a pricing scheme is well performing and might be an easy to communicate alternative.

6.1 Comparison in terms of revenue earned

In this section, we show the simulated revenues of the first five mechanisms for part of the investigated settings. To shorten tables and give a more lucid overview of the study, we decided to cover only a subset of the studied settings.

Table 1: Revenues for $C \leq 20, T = 10$

$\boldsymbol{T = 10}$	$E(\lambda)$	$E(\omega)$	D	S	L
$\boldsymbol{C = 1}$	0.74	0.74	0.74	0.74	0.74
$\boldsymbol{C = 5}$	2.66	2.66	2.67	2.59	2.62
$\boldsymbol{C = 10}$	4.04	4.06	4.06	3.85	3.91
$\boldsymbol{C = 15}$	4.95	5.00	5.00	4.59	4.72
$\boldsymbol{C = 20}$	5.61	5.68	5.68	5.05	5.34

Table 1 shows simulated revenues of all mechanisms with $C \in \{1, 5, 10, 15, 20\}$. For $C = 1$, every mechanism performs (nearly) identical as there is not enough stock to sell more than one unit, and thus, consideration of multiunit purchases is not possible. As we can see, the developed heuristics (mechanisms $E(\cdot)$ and D) are also able to calculate the optimal price for the singleunit case. The more capacity, the higher is the importance of attending customers' demand for more than one unit. This can be seen by comparing mechanisms S and L where the only difference between those mechanisms is that L is aware of customers' multiunit demand while S is ignoring it. Except $C = 1$, L is outperforming S in every setting. This holds all the same for the cases we did not incorporate in this paper. On the other hand, L is dominated by our heuristics. Even though $E(\cdot)$ and D are only heuristics while L numerically finds optimal linear batch prices, the advantage of setting nonlinear batch prices (like $E(\cdot)$ and D do) is overcompensating the incapability to solve optimization problem (9) analytically. Finally, comparing our heuristics to each other, we can observe that $E(\lambda)$ results in the lowest expected revenues while $E(\omega)$ and D are performing nearly identical.

Applying $E(\cdot)$ and D is clearly beneficial in comparison to L and even more to S. As we do not know the optimal solution to optimization problem (9), we cannot test these mechanisms against it. However, to get a feeling for the performance of all mechanisms, we show the simulated revenues relative to an upper bound. Therefore, we additionally simulated revenue for a policy where we were able to observe the realization of the random consumption indicator λ of an arriving customer before quoting the optimal batch prices (cf. Schur 2023). We then divided simulated revenues received from each mechanism by the simulated upper bound received from *PI* and, thereby, get the percentage every mechanism obtains. These percentages are shown in the following figure. In Table 1, we have seen that $E(\omega)$ and D are performing nearly identical. Thus, we opted to only show the line of D representing both mechanisms. We used dotted lines to separate mechanisms that are quoting linear batch prices from the other mechanisms that are following a nonlinear pricing scheme. We let the ordinate start at 0.5 to demonstrate differences between the curves more clearly.

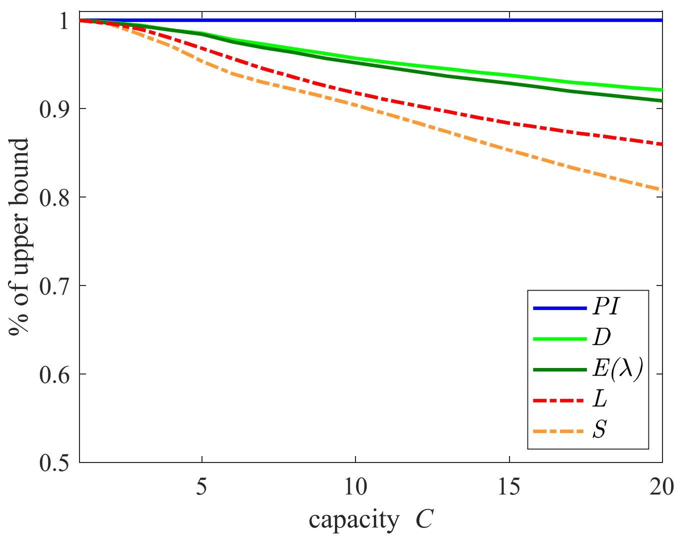


Figure 1: Performance of all mechanism relative to an upper bound for $C \leq 20, T = 10$

Figure 1 shows the same order in the mechanisms for every capacity like Table 1, i.e. $D \geq E \geq L \geq S$. The gap between the upper bound and any other mechanism is increasing in capacity. Probably, this is due to the increasing advantage of knowing the consumption indicator in the upper bound mechanism for higher C. Through full information about customer's consumption indicator and the ability to adapt batch prices to this knowledge, the revenue earned ought to be higher than the (unknown) value of the objective function from optimization problem (9) by a fair amount. This is particularly true whenever there is enough capacity to sell several units to every customer. Thus, we do not assume the gap between upper bound and both heuristics to widen solely because D and $E(\cdot)$ are performing less good for higher capacity. Another indication for this assumption is that *S*, the mechanism where multiunit demand is ignored, is decreasing at a high pace while *D*, *E*, and *L* are becoming quite stable at the end of the curve.

Table 1 already revealed that D and $E(\omega)$ are performing nearly identical even though the idea behind both heuristics is different. However, $E(\lambda)$ is not so far away from these two mechanisms. This could be an indication that all heuristics (especially D and $E(\omega)$) are (strongly) bounded by the optimal solution of optimization problem (9) and, thus, are a good approximation. Finally, we note that our heuristics are performing remarkably better (with a small plus for *D*) than the mechanisms that are most likely applied in practice, L and *S*. Overall, D and $E(\omega)$ are outperforming $E(\lambda)$ particularly for higher capacities.

To get a better impression about the heuristics' performance, we constructed an approximation for the actual (unknown) value of the objective function (9). Therefore, we introduce another mechanism PI^{lin} that, together with *PI* and *L*, provides the basis for approximating $V_t(c)$. In this mechanism, we assume that the firm again can observe next customer's consumption indicator before quoting batch prices. Thereby, we are in line with *PI* except that only linear batch prices are feasible. By comparing PI^{lin} to PI, we can calculate the percentage gain in revenue that results from allowing nonlinear pricing instead of quoting linear batch prices. Although we only can calculate this gain in the case where the firm has full information of next customer's consumption indicator, we assume it also holds for our setting with unknown consumption indicator. By applying this percentage on L, we get an approximation for our nonlinear dynamic pricing approach with unknown consumption indicator and base willingness-to-pay, i.e. for optimization problem (9).

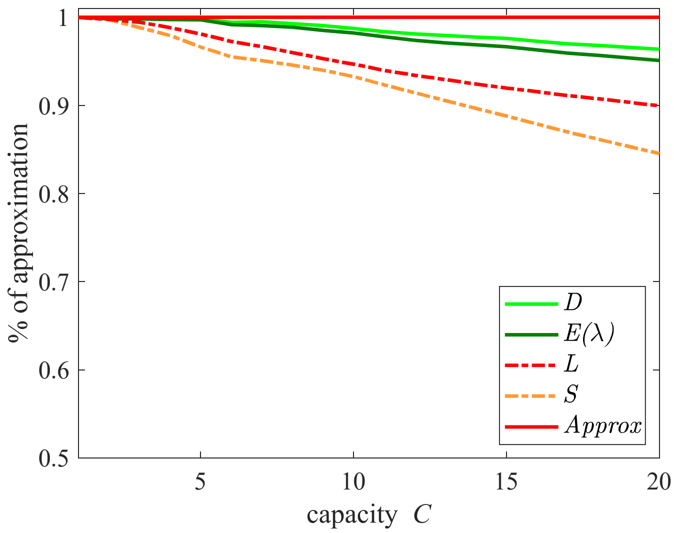
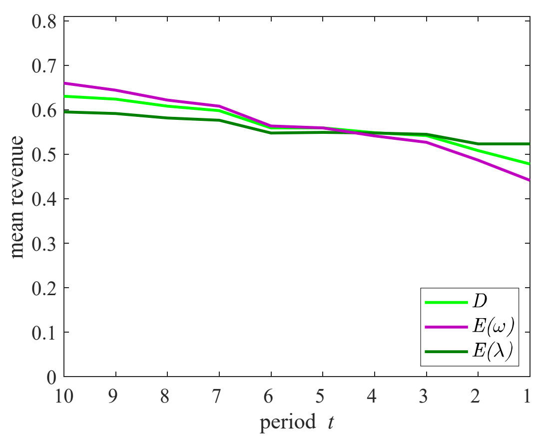


Figure 2: Performance of all mechanism relative to an approximation for the value of the objective function of (9) for $C \leq 20, T = 10$

All heuristics are quite close to the approximation, with D (and $E(\omega)$) always being above 96% of the approximated optimal revenue from (9) (cf. Figure 2). By comparing Figure 2 to Figure 1, we can observe that the curve marking the approximation proceeds in the middle of the upper bound and the outcomes of the heuristics as the gap between the realized percentage and 1 halves. Thus, apart from beating the intuitive mechanisms L and S by a fair amount, they seem to be reasonably close to the real (unknown) optimum.

6.2 Selling strategies of the heuristics

As our heuristics are performing similarly, we want to analyze their strategies. Therefore, we have a closer look to the simulated mean revenue and mean purchases at every point in time, $t = 10, 9, \dots, 1$ starting in state $(T, C) = (10, 20)$. To accomplish this, we tracked purchases and averaged the resulting revenues at every period over all customer streams.



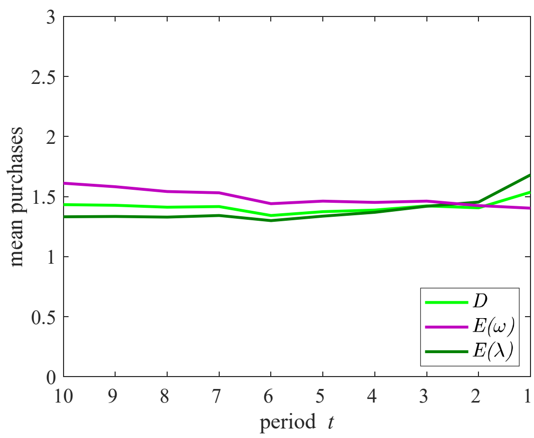


Figure 3: Mean revenues (left) and mean purchases (right) at every point in time, $t = 10, 9, \dots, 1$ with $C = 20$

Although D, $E(\omega)$, and $E(\lambda)$ are performing similarly overall, the strategies seem to be quite different. Figure 3 shows that $E(\lambda)$ is superior to $E(\omega)$ in terms of mean revenue at the end of the selling horizon, but the opposite is true at the beginning. The line representing the strategy of D is running in the middle of the lines representing $E(\omega)$ and $E(\lambda)$. Mean revenues of each heuristic are decreasing over time. While $E(\lambda)$ performs quite stable throughout the whole selling process, the curve representing $E(\omega)$ is declining at a more pronounced rate.

Mean purchases are around 1.5 for every mechanism. The curve resulting from $E(\omega)$ is declining over time, while the curves of the other two mechanisms are quite stable with a sharp, single uptick at the end of the horizon. Over the whole horizon, $E(\omega)$ is selling more units than the other two heuristics. D and $E(\lambda)$ are selling similar quantities, with D being slightly above $E(\lambda)$ overall. This suggests that $E(\omega)$ is pricing units lower than the other two mechanisms.

6.3 Pricing path of *D* and $E(\boldsymbol{\omega})$

We have already seen in Table 1 that D and $E(\omega)$ are resulting in the highest revenues. By examining pricing paths received by these heuristics, we get insights about the structure and behavior of overall well-performing pricing policies. In the figures we present in this section, the jth curve (counting from the bottom) corresponds to the batch price of j units. The starting inventory is $C = 20$ units at period $T = 10$.

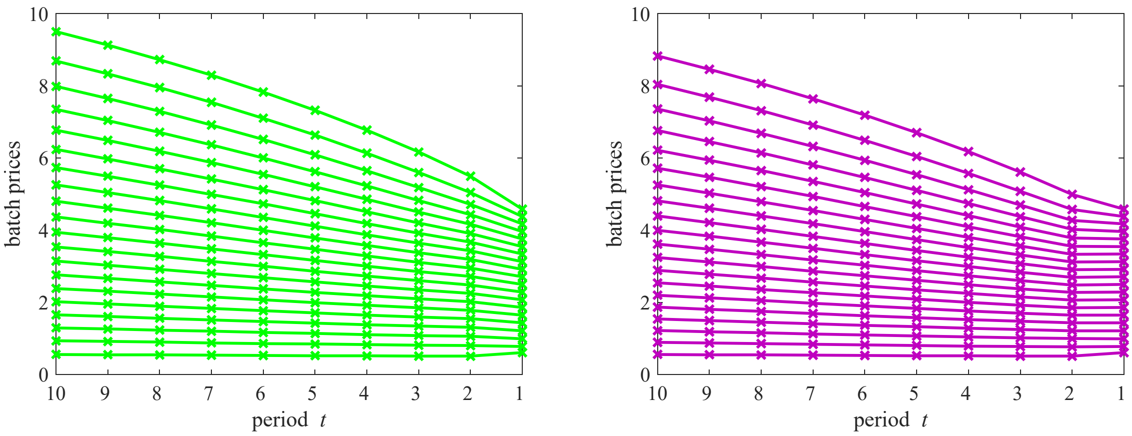


Figure 4: Evolution of batch prices without a purchase for D (left) and E (right) over $t = 10, 9, \dots, 1$ and $C = 20$

In Figure 4, we have depicted the case where no customer wants to purchase anything. Over the whole selling horizon, mechanism D quotes higher prices than mechanism $E(\omega)$. Additionally, prices from both mechanisms are increasing in time with one exception: some batch prices in $t = 1$ are higher than those in $t = 2$. This structural break is a result of enforcing approximate optimality. In $t = 1$, every heuristic applies $\boldsymbol{r}_t^{FA}(c)$ as the solution of the fluid approximation is optimal in the last period (cf. Remark 4). For $t \geq 2$, the maximization picks $\boldsymbol{r}_t^{D}(c)$ and $\boldsymbol{r}_t^{E(\omega)}(c)$ as these prices are resulting in higher revenues than $\boldsymbol{r}_t^{FA}(c)$. The mechanisms to calculate $\boldsymbol{r}_t^{D}(c)$ and $\boldsymbol{r}_t^{E(\omega)}(c)$ are different and still the resulting curves have a similar structure. This indicates an underlying structure good policies have in common.

Batch prices for small numbers of units are virtually linear in batch size (apart from the first unit, the second, third, etc. units cost nearly the same). For large numbers of units, batch prices are convexly increasing in batch size. With a concavely increasing willingness-to-pay curve, these pricing schemes automatically prevent selling a large batch or even the whole stock ($C = 20$) to only one customer. Finally, it is interesting that prices for small batches merely change over time whereas prices for big batches noticeably decrease.

In Figure 3, we have seen that both heuristics result in selling processes where a customer on average purchases approximately 1.5 units. Therefore, we also want to examine the evolution of batch prices in a scenario where alternately two units and one unit are sold, starting with a purchase of 2 units in $t = 10$. As the firm cannot offer batches that are not covered by capacity any longer, most of the curves are ending during the selling horizon.

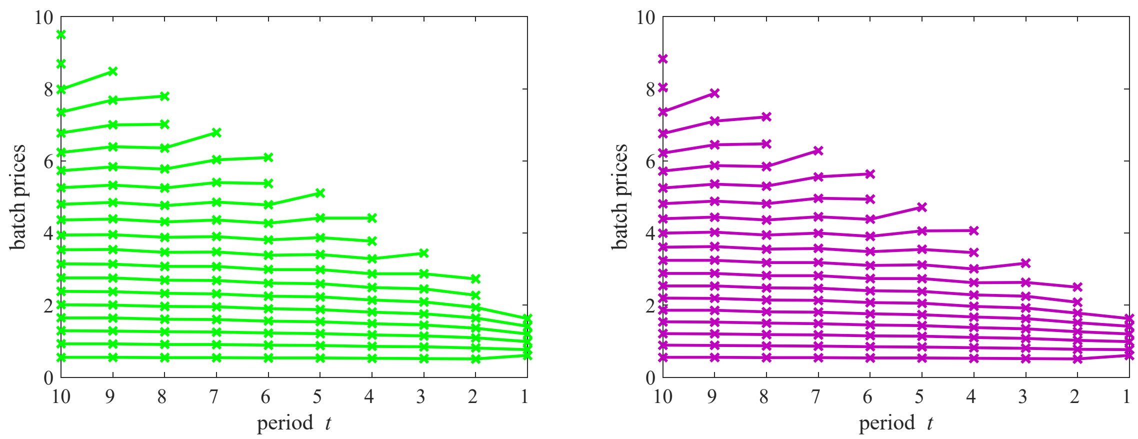


Figure 5: Evolution of batch prices with purchases at every period for D (left) and E (right) over $t = 10, 9, \dots, 1$ and $C = 20$

The pattern the curves draw looks nearly the same for both heuristics. Batch prices obtained by $E(\omega)$ are lower than those obtained by D. The gaps between batch prices are nearly same-sized for smaller batches (again, starting with the two-unit batch) and are increasing for bigger batches. After selling, batch prices for bigger batches increase. This effect is more pronounced after selling two units in comparison to selling one unit. It is a well-observed pricing behavior in (standard) dynamic pricing that prices increase after a sale took place. However, this only holds partially in our multiunit setting as prices for small batches decrease slow and steady over the selling horizon.

To sum up, we have seen two (unusual) effects in Figures 4 and 5: First, prices for small (in comparison to remaining stock) batches are decreasing over time regardless whether a sell takes place or not. This holds for the discussed settings with a reasonably large stock, i.e. $\frac{C}{T} = 2$. For a small stock, i.e. $\frac{T}{C} = 2$, prices for small batches were not always decreasing from one period to the next. To shorten this paper, we excluded the corresponding figures. Second, prices for small and medium batches are increasing approximately linearly in size – starting with the two-unit batch.

6.4 Piecewise-linear pricing

Section 6.3 suggests that the marginal prices of additional units ($j \geq 2$) are nearly constant. To evaluate the loss in revenue if we enforce a piecewise-linear pricing scheme, we implemented mechanism *PL*:

- PL is a mechanism that solves (9) numerically but with an additional constraint that restricts the batch prices to a piecewise-linear pricing scheme with r_1 and $r_j = (j-1) \cdot r_2$ for $j \geq 2$.

This pricing scheme inherits an easy structure, and thus, could be easily applied in practice because a firm has only to quote a price for the first and a price for subsequent units. This applicability and the observations made in Section 6.3 are the main reason why we conducted the following simulation study.

We have seen in Section 6.1 that *D* and $E(\omega)$ are the mechanisms that result in the highest and nearly the same revenues. Thus, we show simulated revenues arising from *PL* relative to those received by *D*. Similar to the approach used for creating Figure 1, we calculate the percentages of *D*'s revenue *PL* obtains. These percentages are shown in the following figure. We let the ordinate start at 0.5 to demonstrate differences between the curves more clearly.

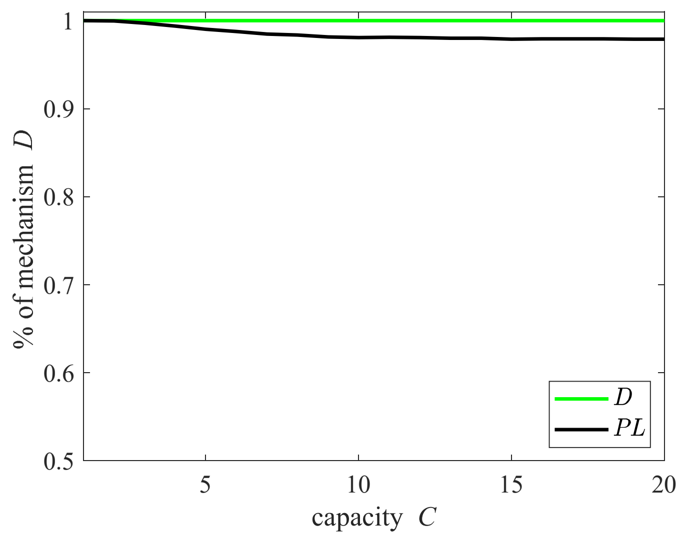


Figure 6: Performance of mechanism PL relative to D for $C \leq 20, T = 10$

Overall, the piecewise-linear pricing scheme performed well. For $C = 1$ and $C = 2$ the results of *D* and *PL* are (as expected) identical. In both cases, there is no actual restrictions on batch prices as there are at most two units that can be sold. For $C \geq 3$ the gap between *D* and *PL* is slowly widening. However, *PL* obtains at least 97.9% of *D*'s revenues.

Apparently, restricting batch prices to a piecewise-linear pricing scheme leads only to a small cutback in revenues. Therefore, it might be favorable in many cases to implement this pricing scheme, as it is easy to communicate and performs very well.

7 Conclusion

In this paper, we present a nonlinear dynamic pricing model. Our model is based on a customer choice model that takes into account customers' base willingness-to-pay and consumption indicator. Although the resulting probability function is complex, we were able to simplify it by using some technical properties and removing unnecessary batch prices from the action space. We proceeded with developing optimality conditions for the stage-wise optimization model and its fluid approximation. However, finding the optimal solution of the stage-wise optimization model remains challenging. To cope with these difficulties, we introduced three novel heuristics: one that uses a decomposition approach and two that calculates expected optimal prices. For every heuristic, we ensured approximate optimality by incorporating the solution of the fluid approximation.

In a simulation study, the heuristics performed very well and quite similar with one of them being only slightly behind the other two. Moreover, they outperformed by a fair amount two approaches that most likely might be applied in practice. We further analyzed both best performing heuristics and found several interesting characteristics of pricing policies: In cases with reasonably large stocks (in our setting with, e.g., $\frac{C}{T} \geq 2$), batch prices for small batches are slowly decreasing over time. This is particularly interesting as it is an indicator that changing prices at a lower rate (not after every customer) might still perform well. This makes the obtained policies also applicable in settings where the firm cannot sustain frequent changes in batch prices due to, e.g., technical reasons, customers' reluctance, or strategic considerations. On the other hand, in cases with a rather short stock (in our setting with, e.g., $\frac{T}{C} \geq 2$), the advantage of nonlinear pricing is declining, whereas a typical (standard) dynamic pricing structure becomes more and more important.

Another finding is that batch prices are nearly linear for low- and medium-sized batches starting with the two-unit batch. This allows an easy to communicate price structure. Instead of displaying a long list, containing prices for every possible batch size, the firm can quote a price for the first unit and then a follow-up price for every additional. Another numerical study verified that this leads only to rather small cutbacks in revenue.

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Supplement: Approximately Optimal Solutions for Nonlinear Dynamic Pricing

S.1 Proof of Lemma 1

We will show the first part of the lemma in detail. The second part follows by a similar argumentation. For every $i \neq k$ with $i, k > j$ and $r_i - r_j \neq 0 \neq r_k - r_j$, it holds that $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m} \Leftrightarrow \frac{r_i - r_j}{r_k - r_j} = \frac{\sum_{m=j}^{i-1} l^m}{\sum_{m=j}^{k-1} l^m}$. W.l.o.g., we assume that $i > k$ (otherwise we could focus on $\frac{r_k - r_j}{r_i - r_j} = \frac{\sum_{m=j}^{k-1} l^m}{\sum_{m=j}^{i-1} l^m}$). We define the function $f(l) = \frac{\sum_{m=j}^{i-1} l^m}{\sum_{m=j}^{k-1} l^m}$ and note that $f(l)$ is continuously differentiable. By differentiating $f(l)$, we show that this function is monotonically increasing on the whole interval $(0, 1)$:

Applying the quotient rule, we note that

$$\frac{d}{dl} f(l) > 0 \Leftrightarrow \sum_{m=j}^{k-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{i-1} l^n - \sum_{m=j}^{i-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{k-1} l^n > 0\,.$$

After some rearrangements of the sums, we get

$$\begin{aligned}
&\sum_{m=j}^{k-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{i-1} l^n - \sum_{m=j}^{i-1} l^m \cdot \frac{d}{dl} \sum_{n=j}^{k-1} l^n = \\
&= \sum_{m=1}^{k-j} \sum_{n=1}^{m} (i-n) l^{k+i-2-m} + \sum_{m=k-j+1}^{i-j} \sum_{n=1}^{k-j} (i+k-j-n-m) l^{k+i-2-m} \\
&+ \sum_{m=i-j+1}^{k+i-1-2j} \sum_{n=m}^{k+i-1-2j} (i+k-1-j-n) l^{k+i-2-m} - \sum_{m=1}^{k-j} \sum_{n=1}^{m} (k-n) l^{k+i-2-m} \\
&- \sum_{m=k-j+1}^{i-j} \sum_{n=1}^{k-j} (k-n) l^{k+i-2-m} - \sum_{m=i-j+1}^{k+i-1-2j} \sum_{n=m}^{k+i-1-2j} (i+k-1-j-n) l^{k+i-2-m} \\
&= \sum_{m=1}^{k-j} \sum_{n=1}^{m} (i-k) l^{k+i-2-m} + \sum_{m=k-j+1}^{i-j} \sum_{n=1}^{k-j} (i-j-m) l^{k+i-2-m} > 0
\end{aligned}$$

for all $l \in (0, 1)$. Together with the independence of $\frac{r_i - r_j}{r_k - r_j}$ from l, it follows that there is at most one $l \in (0,1)$ such that $\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m} = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}$.

To show the second part of this lemma, we use a similar argumentation defining $g(l)=\frac{\sum_{m=k}^{j-1} l^m}{\sum_{m=i}^{j-1} l^m}=1-\frac{\sum_{m=i}^{k-1} l^m}{\sum_{m=i}^{j-1} l^m}$. □

S.2 Proof of Lemma 2

For any $i>j$ it holds that $\max_{0\le k\le j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}\right\}\le\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}=\min_{j+1\le k\le c}\left\{\frac{r_k-r_j}{\sum_{m=j}^{k-1} l^m},1\right\}$ for every $l\in\left[\underline{l}_{ij}^{min}(\boldsymbol{r}),\overline{l}_{ij}^{min}(\boldsymbol{r})\right]$ and $\max_{0\le k\le i-1}\left\{\frac{r_i-r_k}{\sum_{m=k}^{i-1} l^m}\right\}=\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}\le\min_{i+1\le k\le c}\left\{\frac{r_k-r_i}{\sum_{m=i}^{k-1} l^m},1\right\}$ for every $l\in\left[\underline{l}_{ij}^{max}(\boldsymbol{r}),\overline{l}_{ij}^{max}(\boldsymbol{r})\right]$. We show the equality of both intervals by two equivalencies. The first one is given by

$$\max_{0\le k\le j-1}\left\{\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}\right\}\le\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}\Leftrightarrow\max_{0\le k\le j-1}\left\{\frac{r_i-r_k}{\sum_{m=k}^{i-1} l^m}\right\}\le\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}$$

and the second one by

$$\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}=\min_{j+1\le k\le c}\left\{\frac{r_k-r_j}{\sum_{m=j}^{k-1} l^m},1\right\}$$

$$\Leftrightarrow\left(\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}\le\min_{j+1\le k\le i-1}\left\{\frac{r_k-r_j}{\sum_{m=j}^{k-1} l^m}\right\}\right)\wedge\left(\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}\le\min_{i+1\le k\le c}\left\{\frac{r_k-r_j}{\sum_{m=j}^{k-1} l^m},1\right\}\right)$$

$$\Leftrightarrow\left(\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}\ge\max_{j+1\le k\le i-1}\left\{\frac{r_i-r_k}{\sum_{m=k}^{i-1} l^m}\right\}\right)\wedge\left(\frac{r_i-r_j}{\sum_{m=j}^{i-1} l^m}\le\min_{i+1\le k\le c}\left\{\frac{r_k-r_i}{\sum_{m=i}^{k-1} l^m},1\right\}\right).$$

Both equivalencies together result in the proof of the lemma. To verify both equivalencies, the following corollary might help. □

Corollary 1 *For $j>k>i$, one of the following three cases applies:*

1. $\frac{r_k-r_i}{\sum_{m=i}^{k-1} l^m}<\frac{r_j-r_i}{\sum_{m=i}^{j-1} l^m}<\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}$
2. $\frac{r_k-r_i}{\sum_{m=i}^{k-1} l^m}=\frac{r_j-r_i}{\sum_{m=i}^{j-1} l^m}=\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}$
3. $\frac{r_k-r_i}{\sum_{m=i}^{k-1} l^m}>\frac{r_j-r_i}{\sum_{m=i}^{j-1} l^m}>\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}$

Proof: As $\frac{r_j-r_i}{\sum_{m=i}^{j-1} l^m}=\frac{\sum_{m=i}^{k-1} l^m}{\sum_{m=i}^{j-1} l^m}\cdot\frac{r_k-r_i}{\sum_{m=i}^{k-1} l^m}+\frac{\sum_{m=k}^{j-1} l^m}{\sum_{m=i}^{j-1} l^m}\cdot\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}$ is a convex combination of $\frac{r_k-r_i}{\sum_{m=i}^{k-1} l^m}$ and $\frac{r_j-r_k}{\sum_{m=k}^{j-1} l^m}$, the statement above immediately follows. □

S.3 Proof of Lemma 3

To prove the remaining part of Lemma 3, we again assume that our price vector does not follow the given condition, i.e. we choose r_j such that $(r_j - r_{j-1})^{\frac{1}{j-1}} > (r_{j+1} - r_j)^{\frac{1}{j}} \geq 0$. In this case, it holds that $\frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_{j+1} - r_j}{l^j}$ for every $l \in \left[(r_{j+1} - r_j)^{\frac{1}{j}}, 1\right]$ as:

$$\frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_{j+1} - r_j}{l^j} \Leftrightarrow l > \frac{r_{j+1} - r_j}{r_j - r_{j-1}}$$

With $\frac{r_{j+1} - r_j}{r_j - r_{j-1}} = (r_{j+1} - r_j)^{\frac{1}{j}} \cdot \left(\frac{(r_{j+1} - r_j)^{\frac{1}{j}}}{(r_j - r_{j-1})^{\frac{1}{j-1}}}\right)^{j-1} < (r_{j+1} - r_j)^{\frac{1}{j}}$, it holds that the condition of the right side is met for every $l \in \left[(r_{j+1} - r_j)^{\frac{1}{j}}, 1\right]$. As $\frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_{j+1} - r_j}{l^j}$, it holds that

$$w > \frac{r_j - r_{j-1}}{l^{j-1}} \Rightarrow w > \frac{r_{j+1} - r_j}{l^j}$$

and, thus,

$$w \cdot l^{j-1} > r_j - r_{j-1} \Rightarrow w \cdot l^j > r_{j+1} - r_j.$$

This finally results in $w \cdot \sum_{k=0}^{j} l^k - r_{j+1} \geq w \cdot \sum_{k=0}^{j-1} l^k - r_j$ for every $w \in [0,1], l \in \left[(r_{j+1} - r_j)^{\frac{1}{j}}, 1\right]$ such that $w \cdot \sum_{k=0}^{j-1} l^k - r_j = \max_{i \leq j}\left\{w \cdot \sum_{k=0}^{i-1} l^k - r_i\right\}$.

With a similar argumentation, we can conclude that $w \cdot \sum_{k=0}^{j-2} l^k - r_{j-1} \geq w \cdot \sum_{k=0}^{j-1} l^k - r_j$ for every $w \in [0,1], l \in \left[0, (r_{j+1} - r_j)^{\frac{1}{j}}\right]$ such that $w \cdot \sum_{k=0}^{j-1} l^k - r_j = \max_{i \leq j}\left\{w \cdot \sum_{k=0}^{i-1} l^k - r_i\right\}$.

All in all, we can summarize that for every batch size j and a corresponding batch price r_j such that $(r_j - r_{j-1})^{\frac{1}{j-1}} > (r_{j+1} - r_j)^{\frac{1}{j}}$, there is no customer who is willing to purchase j units.

To eliminate demand for j units, the firm could also choose r_j such that $(r_j - r_{j-1})^{\frac{1}{j-1}} = (r_{j+1} - r_j)^{\frac{1}{j}}$. This choice is always possible because of the continuity of prices and $r_{j-1} \leq r_{j+1}$. Thereof, there is no need to considers prices r_j with $(r_j - r_{j-1})^{\frac{1}{j-1}} > (r_{j+1} - r_j)^{\frac{1}{j}}$. □

S.4 Proof of Lemma 4

We recall that $\mathcal{R}_c = \Big\{\boldsymbol{r} \in \mathbb{R}^c : 0 \leq r_1 \leq \cdots \leq r_c \leq c,\ r_j \leq j\ \forall j, (r_j - r_{j-1})^{\frac{1}{j-1}} \leq 1\ for\ j \geq$

$2, and\ (r_j - r_{j-1})^{\frac{1}{j-1}} \le (r_{j+1} - r_j)^{\frac{1}{j}}\ for\ 2 \le j \le c-1\Big\}$ and $\Lambda_j(\boldsymbol{r}) = \Big\{ l \in [0,1] \mid \max_{0\le k\le j-1} \left\{ \frac{r_j - r_k}{\sum_{m=k}^{j-1} l^m} \right\} \le 1 = \min_{j+1\le k\le c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \right\} \Big\} = [\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r})]$.

We start with a special case by observing, that for $\boldsymbol{r} \in \mathcal{R}_c$ it holds by definition that $r_{j+1} = r_j \Rightarrow r_j = r_{j-1}$ and $r_j > r_{j-1} \Rightarrow r_{j+1} > r_j$. Thus, either $r_1 < r_2 < \cdots < r_c$ or there is index i with $r_1 = r_2 = \cdots = r_i < r_{i+1} < \cdots < r_c$.

For the letter case, it holds that $\Lambda_1(\boldsymbol{r}) = \Lambda_2(\boldsymbol{r}) = \cdots = \Lambda_{i-1}(\boldsymbol{r}) = \emptyset$. There is no clear choice of $\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r}),\ j < i$. As $\lambda \ne l$ a.s. for any $l \in [0,1]$, we can as well choose $\underline{l}_j(\boldsymbol{r}), = \bar{l}_j(\boldsymbol{r}) = 0$ (for consistency) even though $[\underline{l}_j(\boldsymbol{r}), \bar{l}_j(\boldsymbol{r})] \ne \emptyset$.

In the following, we will concentrate on j with $r_j < r_{j+1}$ regardless of whether j starts at 1 or at the aforementioned i.

Based on $l \in \Lambda_j(\boldsymbol{r}) \Rightarrow 1 = \min_{j+1\le k\le c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \right\}$, we are interested in the order the lines $f_{kj}(l) = \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}$, $k \ge j+1$, are dropping below 1 for given $\boldsymbol{r} \in \mathcal{R}_c$. As $f_{kj}(l)$ is decreasing in $l \in [0,1]$ this happens at most once on $[0,1]$.

First, we note that $\frac{r_{j+1} - r_j}{l^j} = 1 \Leftrightarrow l = (r_{j+1} - r_j)^{\frac{1}{j}}$ as $r_j < r_{j+1}$. Thereof, we can already conclude from $\boldsymbol{r} \in \mathcal{R}_c$ that $f_{k+1,k}(l)$ drops below 1 before (or at the same time as) $f_{k+2,k+1}(l)$ does, for every $k \ge j$. By Corollary 1, we know that $f_{k+2,k}(l)$ is dropping below 1 between $f_{k+1,k}(l)$ and $f_{k+2,k+1}(l)$. Repeatedly applying Corollary 1 shows that $f_{k+3,k}(l)$ is between $f_{k+2,k}(l)$ and $f_{k+3,k+2}(l)$, $f_{k+4,i}(l)$ is between $f_{k+3,k}(l)$ and $f_{k+4,k+3}(l)$, and so on. We can therefore conclude that the correct order of lines $f_{kj}(l)$, $k > j$, dropping below 1 is $f_{j+1,j}(l)$, $f_{j+2,j}(l), \dots, f_{cj}(l)$. Most importantly, the last $l \in [0,1]$ with $1 = \min_{j+1\le k\le c} \{f_{kj}(l), 1\}$ is the point where $f_{j+1,j}(l) = 1$. At this point $f_{kj}(l) \ge 1$, $k > j$, because of the given order and the fact that every $f_{kj}(l)$ is decreasing in l. This point is given by $l = (r_{j+1} - r_j)^{\frac{1}{j}}$ and we know that the condition $1 = \min_{j+1\le k\le c} \left\{ \frac{r_k - r_j}{\sum_{m=j}^{k-1} l^m}, 1 \right\}$ is met for every $l \le (r_{j+1} - r_j)^{\frac{1}{j}}$. We now can set $\bar{l}_j(\boldsymbol{r}) = (r_{j+1} - r_j)^{\frac{1}{j}}$. For $j = c$, the condition is always met, and we set $\bar{l}_c(\boldsymbol{r}) = 1$.

We now concentrate on $\max_{0\le k\le j-1} \{f_{jk}(l)\} \le 1$. For $j = 1$, this boils down to $r_1 \le 1$ and is always fulfilled for $\boldsymbol{r} \in \mathcal{R}_c$. In this case we set $\underline{l}_1(\boldsymbol{r}) = 0$. If there is an index i with $r_1 = r_2 = \cdots = r_i < r_{i+1} < \cdots < r_c$, then $\max_{0\le k\le i-1} \{f_{ik}(l)\} = 0$. In this case, we set $\underline{l}_i(\boldsymbol{r}) = 0$. For the remaining cases, we again apply Corollary 1. With the same argumentation as above, we can conclude that the correct order of lines $f_{jk}(l)$, $j > k$, dropping below 1 is $f_{j0}(l), f_{j1}(l), \dots, f_{j,j-1}(l)$. Some of these $f_{jk}(l)$, $k < j$, might be below 1 for every $l \in [0,1]$. However, this does not affect the proof of this Lemma. Particularly, it holds that $f_{j,j-1}(l) \leq 1 \Leftrightarrow f_{jk}(l) \leq 1\ \forall k < j \Leftrightarrow \max_{0\leq k\leq j-1}\{f_{jk}(l)\} \leq 1$ because of the given order and the fact that every $f_{jk}(l)$ is decreasing in l. Therefore, the condition $\max_{0\leq k\leq j-1}\{f_{jk}(l)\} \leq 1$ is met for every $l \geq (r_j - r_{j-1})^{\frac{1}{j-1}}$. We now can set $\underline{l}_j(\boldsymbol{r}) = (r_j - r_{j-1})^{\frac{1}{j-1}}$.

To sum up, we have shown that $\underline{l}_1(\boldsymbol{r}) = 0$, $\bar{l}_c(\boldsymbol{r}) = 1$ and $\bar{l}_j(\boldsymbol{r}) = (r_{j+1} - r_j)^{\frac{1}{j}} = \underline{l}_{j+1}(\boldsymbol{r})$ and, thus, the whole lemma. □

S.5 Proof of Lemma 5

Exemplarily, we will walk one of these cases through and assume that $i > j$. The other two would follow in a similar matter and we decided to omit a detailed derivation.

Partially differentiating (5) leads to

$$\frac{d}{d\,r_i}p_j(\boldsymbol{r}) = \frac{d}{d\,r_i}\underline{l}_{j+1}(\boldsymbol{r}) - \frac{d}{d\,r_i}\underline{l}_j(\boldsymbol{r}) + \int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})} \frac{1}{\sum_{m=j}^{i-1} l^m}\,dl$$

$$+ \sum_{k=j+1}^{c}\left(\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m}\cdot\frac{d}{d\,r_i}\bar{l}_{kj}(\boldsymbol{r}) - \frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\underline{l}_{kj}(\boldsymbol{r})\right)^m}\cdot\frac{d}{d\,r_i}\underline{l}_{kj}(\boldsymbol{r})\right)$$

$$- \sum_{k=0}^{j-1}\left(\frac{r_j - r_k}{\sum_{m=k}^{j-1}\left(\bar{l}_{jk}(\boldsymbol{r})\right)^m}\cdot\frac{d}{d\,r_i}\bar{l}_{jk}(\boldsymbol{r}) - \frac{r_j - r_k}{\sum_{m=k}^{j-1}\left(\underline{l}_{jk}(\boldsymbol{r})\right)^m}\cdot\frac{d}{d\,r_i}\underline{l}_{jk}(\boldsymbol{r})\right).$$

With Remark 1, we know that there is $\bar{l}_j^{end}(\boldsymbol{r}) = \max_{k>j}\{\,\bar{l}_{kj}(\boldsymbol{r})\,\} = \max_{k<j}\{\,\bar{l}_{jk}(\boldsymbol{r})\,\}$. We define $k^{max} = \arg\max_{k>j}\{\,\bar{l}_{kj}(\boldsymbol{r})\,\}$ and $k_{max} = \arg\max_{k<j}\{\,\bar{l}_{kj}(\boldsymbol{r})\,\}$. Using the same argumentation as for Remark 2, we can conclude that $\frac{r_{k^{max}} - r_j}{\sum_{m=j}^{k^{max}-1}\left(\bar{l}_{k^{max}j}(\boldsymbol{r})\right)^m}\cdot\frac{d}{d\,r_i}\bar{l}_{k^{max}j}(\boldsymbol{r}) = \frac{r_j - r_{k_{max}}}{\sum_{m=k_{max}}^{j-1}\left(\bar{l}_{jk_{max}}(\boldsymbol{r})\right)^m}\cdot\frac{d}{d\,r_i}\bar{l}_{jk_{max}}(\boldsymbol{r})$.

Moreover, we know from Remark 2, that for every $\bar{l}_{kj}(\boldsymbol{r}) \neq \bar{l}_j^{end}(\boldsymbol{r})$ there is $\underline{l}_{hj}(\boldsymbol{r})$, $h > j$, with

$\frac{r_k - r_j}{\sum_{m=j}^{k-1}\left(\bar{l}_{kj}(\boldsymbol{r})\right)^m} \cdot \frac{d}{d\,r_i}\bar{l}_{kj}(\boldsymbol{r}) - \frac{r_h - r_j}{\sum_{m=j}^{h-1}\left(\underline{l}_{hj}(\boldsymbol{r})\right)^m} \cdot \frac{d}{d\,r_i}\underline{l}_{hj}(\boldsymbol{r}) = 0$. The same is true for $\bar{l}_{jk}(\boldsymbol{r}) \neq \bar{l}_j^{end}(\boldsymbol{r})$ and corresponding $\underline{l}_{jh}(\boldsymbol{r})$, $h < j$. Again with Remark 2, it holds that $\frac{d}{dr_i}\underline{l}_{j+1}(\boldsymbol{r}) - \frac{r_{j+1} - r_j}{\left(\underline{l}_{j+1,j}(\boldsymbol{r})\right)^j} \cdot \frac{d}{dr_i}\underline{l}_{j+1,j}(\boldsymbol{r}) = 0$ and $\frac{r_j - r_{j-1}}{\left(\underline{l}_{j,j-1}(\boldsymbol{r})\right)^{j-1}} \cdot \frac{d}{d\,r_i}\underline{l}_{j,j-1}(\boldsymbol{r}) - \frac{d}{d\,r_i}\underline{l}_j(\boldsymbol{r}) = 0$.

Using all these equations, we get $\frac{d}{d\,r_i}p_j(\boldsymbol{r}) = \int_{\underline{l}_{ij}(\boldsymbol{r})}^{\bar{l}_{ij}(\boldsymbol{r})} \frac{1}{\sum_{m=j}^{i-1} l^m}\,dl$. With similar steps, we can build the partial differentiations for cases $i < j$ and $i = j$. □

S.6 Proof of Lemma 6

If $\bar{l}_j^{end}(\boldsymbol{r}) < \frac{r_{k+1} - r_k}{r_k - r_{k-1}}$ for every j, k with $2 \leq k \leq j$, it holds that $\frac{r_{k+1} - r_k}{l^k} > \frac{r_k - r_{k-1}}{l^{k-1}}$ for every $l \leq \bar{l}_j^{end}(\boldsymbol{r})$ and $2 \leq k \leq j$. By repeatedly applying Corollary 1, it also follows that $\frac{r_{j+1} - r_j}{l^j} > \frac{r_j - r_{j-1}}{l^{j-1}} > \frac{r_j - r_{k-1}}{\sum_{m=k-1}^{j-1} l^m}$ for every $l \leq \bar{l}_j^{end}(\boldsymbol{r})$ and $2 \leq k \leq j$. Thus, $\max_{0 \leq i \leq j-1}\left\{\frac{r_j - r_i}{\sum_{m=i}^{j-1} l^m}\right\} = \max\left\{\frac{r_j - r_{j-1}}{l^{j-1}}, \frac{r_j}{\sum_{m=0}^{j-1} l^m}\right\}$. With the same argumentation, we can also conclude that $\min_{j+1 \leq i \leq c}\left\{\frac{r_i - r_j}{\sum_{m=j}^{i-1} l^m}, 1\right\} = \min\left\{\frac{r_{j+1} - r_j}{l^j}, 1\right\}$. With these two equations, we have shown that batch prices that fulfill the stated conditions lead to a certain structure of the probability function. Thereby, the probability of selling j units is only limited by three events: selling 0, $j - 1$, and $j + 1$ units. □

S.7 Calculation of $p_j^{d,a}(\boldsymbol{r})$

We now only have two possibilities for $\bar{l}_j^{end}(\boldsymbol{r})$ (cf. Remark 1):

1. $\frac{r_{j+1} - r_j}{r_j - r_{j-1}}$

   or

2. l such that $\frac{r_{j+1} - r_j}{l^j} = \frac{r_j}{\sum_{m=0}^{j-1} l^m}$

The lower of these two possible values is $\bar{l}_j^{end}(\boldsymbol{r})$. If the first case applies, it holds that $\Lambda_{j0}^{max}(\boldsymbol{r}) = \emptyset$,

$\Lambda_{j,j-1}^{max}(\boldsymbol{r}) = \left[\underline{l}_j(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$, and $\Lambda_{j+1,j}^{min}(\boldsymbol{r}) = \left[\underline{l}_{j+1}(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$. If the second case applies, it holds that $\Lambda_{j0}^{max}(\boldsymbol{r}) = \left[\bar{l}_{j-1}^{\mathrm{end}}(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$, $\Lambda_{j,j-1}^{max}(\boldsymbol{r}) = \left[\underline{l}_j(\boldsymbol{r}), \bar{l}_{j-1}^{end}(\boldsymbol{r})\right]$, and $\Lambda_{j+1,j}^{min}(\boldsymbol{r}) = \left[\underline{l}_{j+1}(\boldsymbol{r}), \bar{l}_j^{end}(\boldsymbol{r})\right]$. To express the demand function for both cases with one formulation, we bring back the notation $\underline{l}_{ji}^{max}(\boldsymbol{r})$ and $\bar{l}_{ji}^{max}(\boldsymbol{r})$. In the first case, we choose $\underline{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_j(\boldsymbol{r})$ and $\bar{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_{j0}^{max}(\boldsymbol{r}) = \bar{l}_{j0}^{max}(\boldsymbol{r}) = \bar{l}_j^{end}(\boldsymbol{r})$. In the second case, we choose $\underline{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_j(\boldsymbol{r})$, $\bar{l}_{j,j-1}^{max}(\boldsymbol{r}) = \underline{l}_{j0}^{max}(\boldsymbol{r}) = \bar{\mathrm{l}}_{j-1}^{\mathrm{end}}(\boldsymbol{r})$, and $\bar{l}_{j0}^{max}(\boldsymbol{r}) = \bar{l}_j^{end}(\boldsymbol{r})$.

Now, we can write $p_1^{d,a}(\boldsymbol{r}) = -(r_2 - r_1) \cdot \log(r_1)$, $p_2^{d,a}(\boldsymbol{r}) = -(r_3 - r_2) \cdot \left(\frac{1}{\bar{l}_2^{\mathrm{end}}(\boldsymbol{r})} - \frac{1}{\underline{l}_3(\boldsymbol{r})}\right) + \underline{l}_3(\boldsymbol{r}) - \underline{l}_2(\boldsymbol{r}) - r_2 \cdot \left(\log\left(1 + \bar{l}_2^{end}(\boldsymbol{r})\right) - \log\left(1 + \underline{l}_{20}^{max}(\boldsymbol{r})\right)\right) + (r_2 - r_1) \cdot \log(r_1)$, and $p_j^{d,a}(\boldsymbol{r}) = -\frac{r_{j+1} - r_j}{j-1} \cdot \left(\frac{1}{\left(\bar{l}_j^{\mathrm{end}}(\boldsymbol{r})\right)^{j-1}} - \frac{1}{\left(\underline{l}_{j+1}(\boldsymbol{r})\right)^{j-1}}\right) + \underline{l}_{j+1}(\boldsymbol{r}) - \underline{l}_j(\boldsymbol{r}) - \int_{\underline{l}_{j0}^{max}(\boldsymbol{r})}^{\bar{l}_j^{\mathrm{end}}(\boldsymbol{r})} \frac{r_j}{\sum_{m=0}^{j-1} l^m} dl + \frac{r_j - r_{j-1}}{j-2} \cdot \left(\frac{1}{\left(\bar{l}_{j,j-1}^{max}(\boldsymbol{r})\right)^{j-2}} - \frac{1}{\left(\underline{l}_j(\boldsymbol{r})\right)^{j-2}}\right)$.